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Wong Type Oscillation Criteria for Nonlinear Impulsive Differential Equations

S. D. Akgöl[1] | A. Zafer*[2]

[1]Department of Mathematics, Atılım University, İncek, Ankara,, Turkey
[2]College of Engineering and Technology, American University of the Middle East, Egaila, Kuwait

Correspondence
*A. Zafer, College of Engineering and Technology, American University of the Middle East, Egaila, Kuwait. Email: agacik.zafer@aum.edu.kw

Abstract

We present Wong-type oscillation criteria for nonlinear impulsive differential equations having discontinuous solutions and involving both negative and positive coefficients. We use a technique that involves the use of a nonprincipal solution of the associated linear homogeneous equation. The existence of principal and nonpricipal solutions was recently obtained by the present authors in [J. Math. Anal. Appl. 503 (2021) 125311]. As special cases, we have superlinear and sublinear Emden-Fowler equations under impulse effects. It is shown that the oscillation behavior changes due to impulses, in particular impulses acting on the solution itself, not on its derivative. An example is also given to illustrate the importance of the results.

KEYWORDS:
nonlinear; second-order; impulsive differential equation; oscillation; nonprincipal solution

1 | INTRODUCTION.

We start with recalling one of the most celebrated classical oscillation theorem of Wong[1] obtained for linear nonhomogeneous equation

$$(p(t)x')' + q(t)x = f(t). \tag{1}$$

It says that (1) is oscillatory if

$$\limsup_{t\to\infty} \mathcal{H}_0(t) = -\liminf_{t\to\infty} \mathcal{H}_0(t) = \infty,$$

where $\mathcal{H}_0$ is given by

$$\mathcal{H}_0(t) = \int_T^t \frac{1}{p(\tau)v^2(\tau)} \int_T^\tau f(s)v(s)\,\mathrm{d}s\mathrm{d}\tau \tag{2}$$

and v is a positive nonprincipal solution of the corresponding homogeneous equation

$$(p(t)x')' + q(t)x = 0.$$

Recently, the above oscillation theorem has been extended by the present authors in[2] to equations of impulsive type of the form

$$\begin{cases} (p(t)x')' + q(t)x = f(t), & t \neq \theta_i, \\ \Delta x + a_i x = f_i,\ \Delta p(t)x' + b_i x + c_i x' = g_i, & t = \theta_i. \end{cases}$$

Indeed, the first extension was given even earlier in[3] for a much simpler case $a_i = c_i = 0$ as an application of nonprincipal solutions.

A successful generalization of these results to nonlinear differential equations can be found in[4,5]. In particular, related oscillation theorems were derived in[4] for nonlinear equations with mixed type exponents $\beta > 1$ and $\alpha < 1$ of the form

$$(p(t)x')' + q(t)|x|^{\beta-1}x - r(t)|x|^{\alpha-1}x = f(t), \tag{3}$$

where p and q are non-negative functions. Due to the sign changing coefficients the limiting case $\alpha \to 1^-$ and $\beta \to 1^+$ has led an improvement of the Wong oscillation theorem mentioned above. We note that various extensions of Wong type oscillation theorems to nonlinear ordinary differential equations can be found in[7,8,9,10]. However, there is not much work in the literature for their impulsive counterparts[5,11]. On the other hand, there are many results involving different types of oscillation theorems for nonlinear differential equations under impulse effects, see for instance[12,13,14,15] and the references cited therein.

In this work, inspired by the above mentioned studies, we will establish Wong type oscillation criteria for a general class of nonlinear impulsive differential equations of form

$$\begin{cases} (p(t)x')' + q(t)F(x) - r(t)G(x) = f(t), & t \neq \theta_i, \\ \Delta x + a_i F(x) - b_i G(x) = f_i, & t = \theta_i, \\ \Delta p(t)x' + c_i F(x) - d_i G(x) = g_i, & t = \theta_i, \end{cases} \tag{4}$$

where $p > 0, q, r, f$ are piece-wise left continuous functions on $[0, \infty)$; F and G are continuous on $\mathbb{R}$; $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, $\{d_i\}$, $\{f_i\}$ and $\{g_i\}$ are real sequences; $1 - a_i + b_i > 0$; the sequence of impulses $\{\theta_i\}$ is unbounded and increasing. As usual,

$$\Delta y|_{t=\theta_i} = y(\theta_i^+) - y(\theta_i^-)$$

measures the jump of y at $t = \theta_i$.

A function $y(t)$ satisfying (4) is said to be a solution of (4) if it is left continuous on $(\theta_i, \theta_{i+1}]$ and $\lim_{t\to\theta_i^+} y(t)$ exists for each $i \in \mathbb{N}$. As usual, we say that a solution is oscillatory if it is neither eventually positive nor eventually negative.

It turns out that the extension to nonlinear impulsive equations of the form (4) is possible if $\mathcal{H}_0$ in (2) is replaced by

$$\mathcal{H}(t) = \int_a^t \frac{1}{p(s)v^2(s)} \left(\int_a^s \mu(s,\tau) f(\tau) v(\tau)\, \mathrm{d}\tau + \sum_{i=\underline{n}(a)}^{\overline{n}(s)} \mu(s,\theta_i) M_i \right) \mathrm{d}s + \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{f_i}{(1 - a_i + b_i) v(\theta_i)}, \tag{5}$$

where a is an arbitrarily large real number,

$$\underline{n}(t) = \inf\{i \ : \ \theta_i \geq t\}, \quad \overline{n}(t) := \sup\{i \ : \ \theta_i < t\},$$

$$\mu(t,s) = \prod_{i=\underline{n}(s)}^{\overline{n}(t)} (1 - a_i + b_i), \quad t \geq s \geq a,$$

$$M_i = g_i v(\theta_i) - \frac{f_i}{(1 - a_i + b_i)} \Big(p(\theta_i) v'(\theta_i) - (c_i - d_i) v(\theta_i) \Big),$$

and v is a positive nonprincipal solution of

$$\begin{cases} (p(t)x')' + [q(t) - r(t)]x = 0, & t \neq \theta_i, \\ \Delta x + [a_i - b_i]x = 0, \ \Delta p(t)x' + [c_i - d_i]x = 0, & t = \theta_i \end{cases} \tag{6}$$

for $t \geq a$.

The existence of principal and nonprincipal solutions of (6) and their applications to asymptotic integration of certain nonlinear impulsive differential equations can be found in[6].

Obviously, the equation (3) is a special case of (4), and hence the certain particular cases of equation (4) result in the Emden-Fowler type impulsive equations. The corresponding Wong type oscillation theorems will be stated in the present work.

2 | LEMMAS

We give two useful lemmas that we will rely on in the proof of our main oscillation theorem. The first lemma is a construction of an integral equation that is equivalent to given impulsive system (1), and the second one is a simple calculus application.

Let x be a real valued function defined on $[0,\infty)$. We define the operators J_1, J_2, and J_3 as follows:

$$(J_1x)(t) = \int_a^t \mu(t,\tau)\upsilon(\tau)Q(q(\tau),r(\tau),x(\tau))\,\mathrm{d}\tau,$$

$$(J_2x)(t) = \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \mu(t,\theta_i)\left\{ \upsilon(\theta_i)Q(c_i,d_i,x(\theta_i)) - \frac{1}{1-a_i+b_i}\left[p(\theta_i)\upsilon'(\theta_i) - (c_i-d_i)\upsilon(\theta_i)\right]Q(a_i,b_i,x(\theta_i))\right\},$$

and

$$(J_3x)(t) = c_1 + c_2\int_a^t \frac{\mu(s,a)}{p(s)\upsilon^2(s)}\,\mathrm{d}s + \int_a^t \frac{(J_1x)(s)+(J_2x)(s)}{p(s)\upsilon^2(s)}\,\mathrm{d}s + \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{Q(a_i,b_i,x(\theta_i))}{(1-a_i+b_i)\upsilon(\theta_i)}, \tag{7}$$

where

$$Q(a,b,x) = (a-b)x - aF(x) + bG(x).$$

Lemma 1. Let $\upsilon > 0$ be a solution of (6). Then, x is a solution of (4) if and only if it satisfies the integral equation

$$\frac{x}{\upsilon(t)} = J_3x + \mathcal{H}(t), \tag{8}$$

where $\mathcal{H}$ and J_3 are as defined in (5) and (7), respectively.

Proof. Let $x(t)$ be a solution of (8) and $t \neq \theta_l$. It is not difficult to see that

$$(J_3x)' = \frac{1}{p(t)\upsilon^2(t)}(c_2\mu(t,a) + J_1x + J_2x). \tag{9}$$

Thus,

$$p(t)x' = p(t)\upsilon'(t)J_3x + \frac{1}{\upsilon(t)}(c_2\mu(t,a) + J_1x + J_2x) + p(t)z'$$

and so

$$\begin{aligned}(p(t)x')' = (p(t)\upsilon'(t))'J_3x + p(t)\upsilon'(t)(J_3x)' - \frac{\upsilon'(t)}{\upsilon^2(t)}(c_2\mu(t,a) + J_1x + J_2x)\\ + \frac{1}{\upsilon(t)}\left((J_1x)' + (J_2x)'\right) + (p(t)z')',\end{aligned} \tag{10}$$

where $z = \upsilon\mathcal{H}$. One can easily show that

$$(J_1x)' = \mu(t,t)\upsilon(t)Q(q(t),r(t),x), \quad (J_2x)'(t) = 0, \quad \mu(t,t) = 1.$$

Also, as shown in[2] Theorem 2.2, $z = \upsilon\mathcal{H}$ is a solution of

$$\begin{cases} (p(t)z')' + [q(t)-r(t)]z = f(t), & t \neq \theta_i, \\ \Delta z + [a_i - b_i]z = f_i,\ \Delta p(t)z' + [c_i - d_i]z = g_i, & t = \theta_i. \end{cases} \tag{11}$$

Taking the above considerations into account, it follows from (10) that

$$(p(t)x')' = -[q(t)-r(t)]\upsilon(t)J_3x + Q(q(t),r(t),x) - [q(t)-r(t)]z + f(t)$$

or

$$(p(t)x')' + q(t)F(x) - r(t)G(x) = f(t). \tag{12}$$

Now, we look at $t = \theta_l$. Clearly,

$$(J_3x)(\theta_l+) = (J_3x)(\theta_l) + \frac{Q(a_l,b_l,x(\theta_l))}{(1-a_l+b_l)\upsilon(\theta_l)}. \tag{13}$$

Then,

$$\begin{aligned}\Delta x|_{t=\theta_l} &= \Delta[\upsilon(\theta_l)J_3x(\theta_l)] + \Delta z(\theta_l)\\ &= (1-a_l+b_l)\upsilon(\theta_l)J_3x(\theta_l+) - \upsilon(\theta_l)J_3x(\theta_l) - (a_l-b_l)z(\theta_l) + f_l\\ &= -(a_l-b_l)x(\theta_l) + Q(a_l,b_l,x(\theta_l)) + f_l\end{aligned}$$

which implies that

$$\Delta x|_{t=\theta_l} + a_l F(x(\theta_l)) - b_l G(x(\theta_l)) = f_l. \tag{14}$$

Now, it is not difficult to see that $J_1 x(\theta_l+) = (1 - a_l + b_l) J_1 x(\theta_l)$ and

$$\begin{aligned} J_2 x(\theta_l+) =& (1 - a_l + b_l) J_2 x(\theta_l) + (1 - a_l + b_l) v(\theta_l) Q(c_l, d_l, x(\theta_l)) \\ &- \left[p(\theta_l) v'(\theta_l) - (c_l - d_l) v(\theta_l)\right] Q(a_l, b_l, x(\theta_l)). \end{aligned}$$

In view of (9) we can write

$$\begin{aligned} \Delta\left[p(t)v(t)(J_3 x)'(t)\right]|_{t=\theta_l} =& \left[\frac{1 - a_l + b_l}{v(\theta_l+)} - \frac{1}{v(\theta_l)}\right] \left(c_2 \mu(\theta_l, a) + J_1 x(\theta_l) + J_2 x(\theta_l)\right) \\ + \frac{1}{v(\theta_l+)} \Bigg((1 - a_l + b_l) v(\theta_l) Q(c_l, d_l, x(\theta_l)) &- \left[p(\theta_l) v'(\theta_l) - (c_l - d_l) v(\theta_l)\right] Q(a_l, b_l, x(\theta_l)) \Bigg) \\ =& Q(c_l, d_l, x(\theta_l)) - \frac{p(\theta_l) v'(\theta_l) - (c_l - d_l) v(\theta_l)}{(1 - a_l + b_l) v(\theta_l)} Q(a_l, b_l, x(\theta_l)). \end{aligned} \tag{15}$$

On the other hand, using (13) we have

$$\begin{aligned} \Delta\left[p(t)v'(t) J_3 x(t)\right]|_{t=\theta_l} =& \left[p(\theta_l) v'(\theta_l) - (c_l - d_l) v(\theta_l)\right] J_3 x(\theta_l+) - p(\theta_l) v'(\theta_l) J_3 x(\theta_l) \\ =& -(c_l - d_l) v(\theta_l) J_3 x(\theta_l) + \frac{p(\theta_l) v'(\theta_l) - (c_l - d_l) v(\theta_l)}{(1 - a_l + b_l) v(\theta_l)} Q(a_l, b_l, x(\theta_l)). \end{aligned} \tag{16}$$

In view of (15), (16) and the fact that $z(t)$ is a solution of (11) it follows that

$$\begin{aligned} \Delta p(t) x'|_{t=\theta_l} =& \Delta\left[p(\theta_l) v(\theta_l) (J_3 x)'(\theta_l)\right] + \Delta\left[p(\theta_l) v'(\theta_l) J_3 x(\theta_l)\right] + \Delta\left[p(\theta_l) z'(\theta_l)\right] \\ =& Q(c_l, d_l, x(\theta_l)) - (c_l - d_l) v(\theta_l) J_3 x(\theta_l) - (c_l - d_l) z(\theta_l) + g_l, \end{aligned}$$

or

$$\Delta p(t) x'|_{t=\theta_l} + c_l F(x(\theta_l)) - d_l G(x(\theta_l)) = g_l. \tag{17}$$

From (12), (14) and (17), we deduce that $x(t)$ is a solution of (4).

To prove the converse, let x be a solution of (4). Letting $x_1 = x$ and $x_2 = p(t) x_1'$, we may write (4) as a system of first order impulsive differential equations

$$\begin{cases} X' + A(t) X &= \varphi(t, X), \quad t \neq \theta_i \\ \Delta X + A_i X &= \varphi_i(X), \quad t = \theta_i, \end{cases} \tag{18}$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A(t) = \begin{bmatrix} 0 & -1/p(t) \\ q(t) - r(t) & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} a_i - b_i & 0 \\ c_i - d_i & 0 \end{bmatrix},$$

and

$$\varphi(t, X) = \begin{bmatrix} 0 \\ Q(q(t), r(t), x_1) + f(t) \end{bmatrix}, \quad \varphi_i(X) = \begin{bmatrix} Q(a_i, b_i, x_1) + f_i \\ Q(c_i, d_i, x_1) + g_i \end{bmatrix}.$$

As it is shown in[6] Theorem 2.2, the homogeneous equation (6) associated with (4) has principal and nonprincipal solutions u and v such that

$$u(t) = v(t) \int_a^t \frac{\mu(s, a)}{p(s) v^2(s)} \, \mathrm{d}s. \tag{19}$$

Let $\Phi(t, s) = \Phi(t) \Phi^{-1}(s)$, where

$$\Phi(t) = \begin{bmatrix} u(t) & v(t) \\ p(t) u'(t) & p(t) v'(t) \end{bmatrix}.$$

Obviously, $\Phi(t, s)$ is a state transition matrix of

$$\begin{cases} X' + A(t) X &= 0, \quad t \neq \theta_i \\ \Delta X + A_i X &= 0, \quad t = \theta_i. \end{cases}$$

By using the relation (19) one can easily see that the entries a_{11} and a_{12} of $\Phi(t,s)$ are

$$a_{11} = \upsilon(t)\left[-p(s)\upsilon'(s)\int_s^t \frac{\mu(\tau,s)}{p(\tau)\upsilon^2(\tau)}\,\mathrm{d}\tau + \frac{1}{\upsilon(s)}\right], \quad a_{12} = \upsilon(t)\upsilon(s)\int_s^t \frac{\mu(\tau,s)}{p(\tau)\upsilon^2(\tau)}\,\mathrm{d}\tau.$$

Using the variation of parameters formula, we may write from (18) that

$$X(t) = \Phi(t,a)X(a) + \int_a^t \Phi(t,s)\varphi(s,X)\,\mathrm{d}s + \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \Phi(t,\theta_i+)\varphi_i(X).$$

Therefore,

$$\begin{aligned}
x_1(t) =& c_1\upsilon(t) + c_2\upsilon(t)\int_a^t \frac{\mu(s,a)}{p(s)\upsilon^2(s)}\,\mathrm{d}s \\
&+ \upsilon(t)\Bigg\{\int_a^t \big[Q(q(s),r(s),x_1(s)) + f(s)\big]\upsilon(s)\int_s^t \frac{\mu(\tau,s)}{p(\tau)\upsilon^2(\tau)}\,\mathrm{d}\tau\mathrm{d}s && (20)\\
&- \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \big[Q(a_i,b_i,x_1(\theta_i)) + f_i\big]\left[p(\theta_i+)\upsilon'(\theta_i+)\int_{\theta_i}^t \frac{\mu(\tau,\theta_i+)}{p(\tau)\upsilon^2(\tau)}\,\mathrm{d}\tau - \frac{1}{\upsilon(\theta_i+)}\right] && (21)\\
&+ \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \big[Q(c_i,d_i,x_1(\theta_i)) + g_i\big]\upsilon(\theta_i+)\int_{\theta_i}^t \frac{\mu(\tau,\theta_i+)}{p(\tau)\upsilon^2(\tau)}\,\mathrm{d}\tau\Bigg\}. && (22)
\end{aligned}$$

By changing the order of integration in (20), the order of integration and summation both in (21) and in (22), and using $\mu(t,\theta_i+) = \mu(t,\theta_i)/(1-a_i+b_i)$, we finally obtain that x_1 is a solution of (8). □

We will also need the following simple lemma. The proof is elementary, so we omit it.

Lemma 2. Suppose that $xF(x) > 0$, $xG(x) > 0$ for $x \neq 0$. If

$$\lim_{|x|\to\infty}\frac{F(x)}{x} > 1,\ \lim_{|x|\to 0}\frac{F(x)}{x} < 1, \quad \lim_{|x|\to\infty}\frac{G(x)}{x} < 1,\ \lim_{|x|\to 0}\frac{G(x)}{x} > 1, \tag{23}$$

then

$$F_m = -\min_{x\le 0}\big[x - F(x)\big],\ F_M = \max_{x\ge 0}\big[x - F(x)\big],\ G_m = -\min_{x\ge 0}\big[x - G(x)\big],\ G_M = \max_{x\le 0}\big[x - G(x)\big]$$

exist as positive real numbers.

3 | WONG'S OSCILLATION THEOREM

In addition to $\mathcal{H}$ given by (5), we define

$$\begin{aligned}
\overline{N}(t) :=& \int_a^t \frac{1}{p(s)\upsilon^2(s)}\Bigg\{\int_a^s \mu(s,\tau)\upsilon(\tau)\big[q(\tau)F_M + r(\tau)G_m\big]\,\mathrm{d}\tau + \sum_{i=\underline{n}(a)}^{\overline{n}(s)} \mu(s,\theta_i)\big[C_iF_M + D_iG_m\big] \\
&+ \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{1}{(1-a_i+b_i)\upsilon(\theta_i)}\big[a_iF_M + b_iG_m\big]
\end{aligned}$$

and

$$\begin{aligned}
\underline{N}(t) :=& \int_a^t \frac{1}{p(s)\upsilon^2(s)}\Bigg\{\int_a^s \mu(s,\tau)\upsilon(\tau)\big[q(\tau)F_m + r(\tau)G_M\big]\,\mathrm{d}\tau + \sum_{i=\underline{n}(a)}^{\overline{n}(s)} \mu(s,\theta_i)\Big[C_iF_m + D_iG_M\Big]\Bigg\}\,\mathrm{d}s \\
&+ \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{1}{(1-a_i+b_i)\upsilon(\theta_i)}\big[a_iF_m + b_iG_M\big],
\end{aligned}$$

where F_m, F_M, G_m, and G_M are as in Lemma 2, and

$$C_i := c_i v(\theta_i) - \frac{a_i}{1-a_i+b_i}\Big[p(\theta_i)v'(\theta_i) - (c_i - d_i)v(\theta_i)\Big],$$

$$D_i := d_i v(\theta_i) - \frac{b_i}{1-a_i+b_i}\Big[p(\theta_i)v'(\theta_i) - (c_i - d_i)v(\theta_i)\Big].$$

We may now state and prove our main result for oscillation of (4).

Theorem 1. Let (6) be nonoscillatory and $v(t) > 0$ be its nonprincipal solution. Suppose that (23) holds, and $q(t)$, $r(t)$, a_i, b_i, C_i and D_i are non-negative. If

$$\limsup_{t\to\infty}[\mathcal{H}(t) - \underline{N}(t)] = -\liminf_{t\to\infty}[\mathcal{H}(t) + \overline{N}(t)] = \infty, \tag{24}$$

then the impulsive equation (4) is oscillatory.

Proof. Let $x(t)$ be a solution of the nonlinear impulsive equation (4). By Lemma 1, $x(t)$ satisfies (8). We need to estimate $J_3 x$ given in (7). To do so, we need to estimate $J_1 x$ and $J_2 x$. Since $q(t)$ and $r(t)$ are non-negative, in view of Lemma 2, we can write from

$$(J_1 x)(t) = \int_a^t \mu(t,\tau)v(\tau)\Big(q(\tau)\big[x - F(x)\big] - r(\tau)\big[x - G(x)\big]\Big)\,\mathrm{d}\tau$$

that

$$(J_1 x)(t)\begin{cases} \le & \int_a^t \mu(t,\tau)v(\tau)\Big(q(\tau)F_M + r(\tau)G_m\Big)\,\mathrm{d}\tau, \quad x > 0,\\ \ge & \int_a^t \mu(t,\tau)v(\tau)\Big(q(\tau)F_m + r(\tau)G_M\Big)\,\mathrm{d}\tau, \quad x < 0.\end{cases} \tag{25}$$

Next, we rewrite $(J_2 x)(t)$ as

$$\begin{aligned}(J_2 x)(t) = \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \mu(t,\theta_i)\Big\{ & v(\theta_i)\Big((c_i - d_i)x(\theta_i)) - c_i F(x(\theta_i)) + d_i G(x(\theta_i))\Big)\\ & - \frac{1}{1-a_i+b_i}\Big[p(\theta_i)v'(\theta_i) - (c_i-d_i)v(\theta_i)\Big]\Big((a_i - b_i)x(\theta_i)) - a_i F(x(\theta_i)) + b_i G(x(\theta_i))\Big)\Big\}\\ = \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \mu(t,\theta_i)\Big\{ & \Big(C_i[x - F(x)] - D_i[x - G(x)]\Big)\Big\}.\end{aligned}$$

Since a_i, b_i, C_i and D_i are non-negative, by Lemma 2 we easily obtain

$$(J_2 x)(t)\begin{cases} \le & \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \mu(t,\theta_i)\Big(C_i F_M + D_i G_m\Big), \quad x > 0\\ \ge & \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \mu(t,\theta_i)\Big(C_i F_m + D_i G_M\Big), \quad x < 0.\end{cases} \tag{26}$$

On the other hand, it is not difficult to see that

$$\sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{Q(a_i,b_i,x(\theta_i))}{(1-a_i+b_i)v(\theta_i)}\begin{cases} \le & \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{1}{(1-a_i+b_i)v(\theta_i)}\Big(a_i F_M + b_i G_m\Big), \quad x > 0,\\ \ge & \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{1}{(1-a_i+b_i)v(\theta_i)}\Big(a_i F_m + b_i G_M\Big), \quad x < 0.\end{cases} \tag{27}$$

Employing (25), (26), and (27) in (7) leads to

$$(J_3x)(t)\begin{cases}\leq \quad c_1 + c_2 \int\limits_a^t \dfrac{\mu(s,a)}{p(s)v^2(s)}\,\mathrm{d}s + \overline{N}(t), & x > 0\\ \geq \quad c_1 + c_2 \int\limits_a^t \dfrac{\mu(s,a)}{p(s)v^2(s)}\,\mathrm{d}s - \underline{N}(t), & x < 0.\end{cases} \tag{28}$$

By using the estimate (28) in (8), we easily obtain

$$\frac{x(t)}{v(t)} \leq c_1 + c_2 \int\limits_a^t \frac{\mu(s,a)}{p(s)v^2(s)}\,\mathrm{d}s + \mathcal{H}(t) + \overline{N}(t), \quad x > 0, \tag{29}$$

$$\frac{x(t)}{v(t)} \geq c_1 + c_2 \int\limits_a^t \frac{\mu(s,a)}{p(s)v^2(s)}\,\mathrm{d}s + \mathcal{H}(t) - \underline{N}(t), \quad x < 0. \tag{30}$$

By making use of (24) and the fact that

$$\int\limits_a^t \frac{\mu(s,a)}{p(s)v^2(s)}\,\mathrm{d}s < \infty,$$

it follows from (29) and (30) that

$$\limsup_{t\to\infty} \frac{x(t)}{v(t)} = -\liminf_{t\to\infty} \frac{x(t)}{v(t)} = \infty.$$

Since $v(t) > 0$, we conclude that $x(t)$ must be oscillatory. □

In the next section we illustrate some important special cases of Theorem 1.

4 | EMDEN-FOWLER TYPE IMPULSIVE EQUATIONS

Let $F(x) = |x|^{\beta-1}x$ and $G(x) = |x|^{\alpha-1}x$, where $0 < \alpha < 1 < \beta$, then (4) turns into the Emden-Fowler type impulsive equation with superlinear and sublinear terms

$$\begin{cases}(p(t)x')' + q(t)|x|^{\beta-1}x - r(t)|x|^{\alpha-1}x = f(t), & t \neq \theta_i,\\ \Delta x + a_i|x|^{\beta-1}x - b_i|x|^{\alpha-1}x = f_i, & t = \theta_i,\\ \Delta p(t)x' + c_i|x|^{\beta-1}x - d_i|x|^{\alpha-1}x = g_i, & t = \theta_i.\end{cases} \tag{31}$$

By a simple calculation we have

$$F_m = F_M = \beta^{\beta/(1-\beta)}(\beta - 1), \quad G_m = G_M = \alpha^{\alpha/(1-\alpha)}(1-\alpha).$$

Therefore, we obtain the following oscillation theorem.

Theorem 2. Let (6) be nonoscillatory and $v(t) > 0$ be its nonprincipal solution. Suppose that $q(t)$, $r(t)$, a_i, b_i, C_i and D_i are non-negative. Then, the halflinear impulsive Emden-Fowler equation (31) is oscillatory provided that

$$\limsup_{t\to\infty}[\mathcal{H}(t) - N(t)] = -\liminf_{t\to\infty}[\mathcal{H}(t) + N(t)] = \infty, \tag{32}$$

where $\mathcal{H}(t)$ is as given in (5) and

$$N(t) := \int_a^t \frac{1}{p(s)v^2(s)} \left\{ \int_a^s \mu(s,\tau)v(\tau)\big[q(\tau)\beta^{\beta/(1-\beta)}(\beta-1) + r(\tau)\alpha^{\alpha/(1-\alpha)}(1-\alpha)\big]\,\mathrm{d}\tau \right.$$
$$\left. + \sum_{i=\underline{n}(a)}^{\overline{n}(s)} \mu(s,\theta_i)\Big[C_i\beta^{\beta/(1-\beta)}(\beta-1) + D_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)\Big] \right\} \mathrm{d}s$$
$$+ \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{1}{(1-a_i+b_i)v(\theta_i)}\big[a_i\beta^{\beta/(1-\beta)}(\beta-1) + b_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)\big].$$

Taking $\alpha \to 1^-$ and $\beta \to 1^+$ in (31) results in a linear impulsive equation

$$\begin{cases} (p(t)x')' + [q(t) - r(t)]x = f(t), & t \neq \theta_i, \\ \Delta x + [a_i - b_i]x = f_i, & t = \theta_i, \\ \Delta p(t)x' + [c_i - d_i]x = g_i, & t = \theta_i. \end{cases} \tag{33}$$

In this case, $\alpha^{\alpha/(1-\alpha)} \to 1/e$ and $\beta^{\beta/(1-\beta)} \to 1/e$, and so $F_m = F_M = G_m = G_M = 0$. Thus we easily obtain the following corollary.

Corollary 1. Let (6) be nonoscillatory and $v(t) > 0$ be its nonprincipal solution. Then, the linear impulsive Emden-Fowler equation (33) is oscillatory provided that

$$\limsup_{t\to\infty} \mathcal{H}(t) = -\liminf_{t\to\infty} \mathcal{H}(t) = \infty,$$

where $\mathcal{H}(t)$ is as given in (5).

We finally state the Wong type oscillation theorems for the superlinear impulsive Emden-Fowler type equation

$$\begin{cases} (p(t)x')' + q(t)|x|^{\beta-1}x = f(t), & t \neq \theta_i, \\ \Delta x + a_i|x|^{\beta-1}x = f_i, & t = \theta_i, \\ \Delta p(t)x' + c_i|x|^{\beta-1}x = g_i, & t = \theta_i \end{cases} \tag{34}$$

and the sublinear impulsive Emden-Fowler type equation

$$\begin{cases} (p(t)x')' - r(t)|x|^{\alpha-1}x = f(t), & t \neq \theta_i, \\ \Delta x - b_i|x|^{\alpha-1}x = f_i, & t = \theta_i, \\ \Delta p(t)x' - d_i|x|^{\alpha-1}x = g_i, & t = \theta_i. \end{cases} \tag{35}$$

The following results are readily available from Theorem 2.

Corollary 2. Let the homogeneous equation associated with (34) be nonoscillatory and $v(t) > 0$ be its nonprincipal solution. Suppose that $q(t)$, a_i and $C_{i1} := c_iv(\theta_i) - a_ip(\theta_i)v'(\theta_i)$ are non-negative. Then, the superlinear impulsive Emden-Fowler equation (34) under impulse effects is oscillatory provided that

$$\limsup_{t\to\infty}[\mathcal{H}_1(t) - N_1(t)] = -\liminf_{t\to\infty}[\mathcal{H}_1(t) + N_1(t)] = \infty,$$

where

$$\mathcal{H}_1(t) := \int_a^t \frac{1}{p(s)v^2(s)} \left(\int_a^s \prod_{i=\underline{n}(\tau)}^{\overline{n}(s)} (1-a_i)f(\tau)v(\tau)\,\mathrm{d}\tau + \sum_{i=\underline{n}(a)}^{\overline{n}(s)} \prod_{k=i}^{\overline{n}(s)} (1-a_k)M_{i1} \right) \mathrm{d}s + \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{f_i}{(1-a_i)v(\theta_i)},$$

with

$$M_{i1} := g_iv(\theta_i) - \frac{f_i}{(1-a_i)}\Big(p(\theta_i)v'(\theta_i) - c_iv(\theta_i)\Big),$$

$$N_1(t) := \int_a^t \frac{1}{p(s)v^2(s)} \left\{ \int_a^s \prod_{i=\underline{n}(\tau)}^{\overline{n}(s)} (1-a_i)v(\tau)q(\tau)\beta^{\beta/(1-\beta)}(\beta-1)\,\mathrm{d}\tau \right.$$
$$\left. + \sum_{i=\underline{n}(a)}^{\overline{n}(s)} \left[\prod_{k=i+1}^{\overline{n}(s)} (1-a_k)C_{i1}\beta^{\beta/(1-\beta)}(\beta-1) \right] \right\} \mathrm{d}s + \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{a_i}{(1-a_i)v(\theta_i)}\beta^{\beta/(1-\beta)}(\beta-1).$$

Corollary 3. Let the homogeneous equation associated with (35) be nonoscillatory and $v(t) > 0$ be its nonprincipal solution. Suppose that $r(t)$, b_i and $D_{i2} := d_i v(\theta_i) - b_i p(\theta_i)v'(\theta_i)$ are non-negative. Then, the sublinear Emden-Fowler equation (35) under impulse effects is oscillatory provided that

$$\limsup_{t\to\infty}[\mathcal{H}_2(t) - N_2(t)] = -\liminf_{t\to\infty}[\mathcal{H}_2(t) + N_2(t)] = \infty,$$

where

$$\mathcal{H}_2(t) := \int_a^t \frac{1}{p(s)v^2(s)} \left(\int_a^s \prod_{i=\underline{n}(\tau)}^{\overline{n}(s)} (1+b_i)f(\tau)v(\tau)\,\mathrm{d}\tau + \sum_{i=\underline{n}(a)}^{\overline{n}(s)} \prod_{k=i}^{\overline{n}(s)} (1+b_k)M_{i2} \right) \mathrm{d}s + \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{f_i}{(1+b_i)v(\theta_i)},$$

with

$$M_{i2} := g_i v(\theta_i) - \frac{f_i}{(1+b_i)}\Big(p(\theta_i)v'(\theta_i) + d_i v(\theta_i)\Big),$$

$$N_2(t) := \int_a^t \frac{1}{p(s)v^2(s)} \left\{ \int_a^s \prod_{i=\underline{n}(\tau)}^{\overline{n}(s)} (1+b_i)v(\tau)r(\tau)\alpha^{\alpha/(1-\alpha)}(1-\alpha)\,\mathrm{d}\tau \right.$$
$$\left. + \sum_{i=\underline{n}(a)}^{\overline{n}(s)} \left[\prod_{k=i+1}^{\overline{n}(s)} (1+b_k)D_{i2}\alpha^{\alpha/(1-\alpha)}(1-\alpha) \right] \right\} \mathrm{d}s + \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{b_i}{(1+b_i)v(\theta_i)}\alpha^{\alpha/(1-\alpha)}(1-\alpha).$$

5 | EXAMPLE

In this section we provide an example to illustrate the efficiency of Theorem 2. So, we consider an Emden-Fowler type impulsive equation with superlinear and sublinear terms

$$\begin{cases} (t^2(t^2+1)x')' + 4|x|^{\beta-1}x - 10(t^2+1)|x|^{\alpha-1}x = \sin t, & t \neq i, \quad t \geq 1, \\ \Delta x + i|x|^{\beta-1}x - \dfrac{i^2+1}{i}|x|^{\alpha-1}x = (-1)^i i^3(i+1), & t = i, \\ \Delta(t^2(t^2+1)x') + 2i^2(i^2+1)|x|^{\beta-1}x - 2(i^2+1)^2|x|^{\alpha-1}x = 2i, & t = i, \quad i > 1. \end{cases} \tag{36}$$

Comparing with the general form (31), we observe that $\theta_i = i$, $p(t) = t^2(t^2+1)$, $q(t) = 4$, $r(t) = 10(t^2+1)$, $a_i = i$, $b_i = (i^2+1)/i$, $c_i = 2i^2(i^2+1)$, $d_i = 2(i^2+1)^2$, $f(t) = \sin t$, $f_i = (-1)^i i^3(i+1)$ and $g_i = 2i$. Thus, $1 - a_i + b_i = (i+1)/i > 0$,

$$C_i = c_i v(\theta_i) - \frac{a_i}{1-a_i+b_i}\Big[p(\theta_i)v'(\theta_i) - (c_i - d_i)v(\theta_i)\Big] = 0$$

and

$$D_i = d_i v(\theta_i) - \frac{b_i}{1-a_i+b_i}\Big[p(\theta_i)v'(\theta_i) - (c_i - d_i)v(\theta_i)\Big] = 0,$$

where $v(t) = it^2$, $t \in (i-1, i]$ is a nonprincipal solution of the associated homogeneous equation

$$\begin{cases} (t^2(t^2+1)x')' + 4x - 10(t^2+1)x = 0, \quad t \neq i, \quad t \geq 1, \\ \Delta x + ix - \dfrac{i^2+1}{i}x = 0, \quad \Delta(t^2(t^2+1)x') + 2i^2(i^2+1)x - 2(i^2+1)^2x = 0, \quad t = i, \quad i > 1. \end{cases}$$

For convenience we write

$$\begin{aligned}
N(t) =& \int_1^t \frac{1}{p(s)v^2(s)}\big(y_1(s)+y_2(s)\big)\,\mathrm{d}s \\
&+ \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{1}{(1-a_i+b_i)v(\theta_i)}\big[a_i\beta^{\beta/(1-\beta)}(\beta-1)+b_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)\big], \\
\mathcal{H}(t) =& y_3(t)+y_4(t),
\end{aligned}$$

where

$$\begin{aligned}
y_1(t) :=& \int_a^t \mu(t,\tau)v(\tau)\big[q(\tau)\beta^{\beta/(1-\beta)}(\beta-1)+r(\tau)\alpha^{\alpha/(1-\alpha)}(1-\alpha)\big]\,\mathrm{d}\tau, \\
y_2(t) :=& \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \mu(t,\theta_i)\Big[C_i\beta^{\beta/(1-\beta)}(\beta-1)+D_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)\Big], \\
y_3(t) :=& \int_a^t \frac{1}{p(s)v^2(s)}\left(\int_a^s \mu(s,\tau)f(\tau)v(\tau)\,\mathrm{d}\tau + \sum_{i=\underline{n}(a)}^{\overline{n}(s)} \mu(s,\theta_i)M_i\right)\mathrm{d}s, \\
y_4(t) :=& \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{f_i}{(1-a_i+b_i)v(\theta_i)}.
\end{aligned}$$

Let $a=1$, $t\in(k-1,k]$ and $s\in(j-1,j]$ for $k>j>2$. It follows that

$$\mu(s,\tau)=(\overline{n}(s)+1)/\underline{n}(\tau)=(j+1)/\underline{n}(\tau),$$

and so, $\mu(s,i)=(j+1)/i$, $\underline{n}(\tau)=i-1$ for $\tau\in(i-1,i]$ and $\underline{n}(\tau)=j-1$ for $\tau\in(j-1,s)$. Thus, we have

$$\begin{aligned}
y_1(s) =& (j+1)\sum_{i=2}^{j-1}\frac{i}{i-1}\int_{i-1}^{i}\tau^2\Big[4\beta^{\beta/(1-\beta)}(\beta-1)+10(\tau^2+1)\alpha^{\alpha/(1-\alpha)}(1-\alpha)\Big]\,\mathrm{d}\tau \\
&+\frac{(j+1)j}{j-1}\int_{j-1}^{s}\tau^2\Big[4\beta^{\beta/(1-\beta)}(\beta-1)+10(\tau^2+1)\alpha^{\alpha/(1-\alpha)}(1-\alpha)\Big]\,\mathrm{d}\tau \\
=& O(s^6), \quad s\to\infty,
\end{aligned}$$

$$y_2(s)=\sum_{i=2}^{j}\frac{j+1}{i}\big[C_i\beta^{\beta/(1-\beta)}(\beta-1)+D_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)\big)\big]=0,$$

and

$$\begin{aligned}
&\sum_{i=\underline{n}(a)}^{\overline{n}(t)}\frac{1}{(1-a_i+b_i)v(\theta_i)}\big[a_i\beta^{\beta/(1-\beta)}(\beta-1)+b_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)\big] \\
&=\sum_{i=2}^{\overline{n}(t)}\frac{i}{(i+1)i^3}\Big[i\beta^{\beta/(1-\beta)}(\beta-1)+\frac{i^2+1}{i}\alpha^{\alpha/(1-\alpha)}(1-\alpha)\Big] \\
&=O(1), \quad t\to\infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}N(t) =&\sum_{i=2}^{k-1}\int_{i-1}^{i} \frac{1}{s^6(s^2+1)i^2}\big(y_1(s)+y_2(s)\big)\,\mathrm{d}s+\int_{k-1}^{t}\frac{1}{s^6(s^2+1)k^2}\big(y_1(s)+y_2(s)\big)\,\mathrm{d}s\\&+\sum_{i=\underline{n}(a)}^{\overline{n}(t)}\frac{1}{(1-a_i+b_i)v(i)}\big[a_i\beta^{\beta/(1-\beta)}(\beta-1)+b_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)\big]\\=&O(1),\quad t\to\infty.\end{aligned}\tag{37}$$

On the other hand, we calculate that $M_i = 2i^4\big[1-(-1)^i i^3(i+1)(i^2+1)\big]$,

$$\begin{aligned}\int_a^s \mu(s,\tau)f(\tau)v(\tau)\,\mathrm{d}\tau =&(j+1)\sum_{i=2}^{j-1}\frac{i}{i-1}\int_{i-1}^{i}\tau^2\sin\tau\,\mathrm{d}\tau+\frac{(j+1)j}{j-1}\int_{j-1}^{s}\tau^2\sin\tau\,\mathrm{d}\tau\\=&O(s^4),\quad s\to\infty,\end{aligned}\tag{38}$$

and

$$\begin{aligned}\sum_{i=\underline{n}(a)}^{\overline{n}(s)}\mu(s,\theta_i)M_i =& \sum_{i=2}^{j}\frac{j+1}{i}2i^4\big[1-(-1)^i i^3(i+1)(i^2+1)\big]\\=&(-1)^{j+1}j^{10}+O(j^9),\quad j\to\infty.\end{aligned}\tag{39}$$

Then, from (38) we have

$$\begin{aligned}\int_a^t\frac{1}{p(s)v^2(s)}\int_a^s\mu(s,\tau)f(\tau)v(\tau)\,\mathrm{d}\tau =& \sum_{j=2}^{k-1}\int_{j-1}^{j}\frac{1}{j^2s^6(s^2+1)}\int_1^s\mu(s,\tau)f(\tau)v(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\\&+\int_{k-1}^{t}\frac{1}{k^2s^6(s^2+1)}\int_1^s\mu(s,\tau)f(\tau)v(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\\=&O\big(1\big),\quad t\to\infty.\end{aligned}\tag{40}$$

In a similar way, (39) implies that

$$\int_a^t\frac{1}{p(s)v^2(s)}\sum_{i=\underline{n}(a)}^{\overline{n}(s)}\mu(s,\theta_i)M_i\,\mathrm{d}s = (-1)^k\frac{k^8}{7t^7}+O(1),\quad t\to\infty.\tag{41}$$

Finally,

$$y_4(t)=\sum_{i=2}^{k}(-1)^i i = (-1)^k\frac{k+2}{4}+\frac{3}{4}.\tag{42}$$

The sum of (40), (41) and (42) yields to $\limsup_{t\to\infty}\big[y_3(t)+y_4(t)\big] = -\liminf_{t\to\infty}\big[y_3(t)+y_4(t)\big]=\infty$, or

$$\limsup_{t\to\infty}\mathcal{H}(t) = -\liminf_{t\to\infty}\mathcal{H}(t)=\infty.\tag{43}$$

Combining (37) and (43), we see that (32) holds. Since all the conditions of Theorem 2 are satisfied, we conclude that (36) is oscillatory. The oscillation behavior of the solution when $\alpha = 1/2$ and $\beta = 3/2$ is shown in Figure 1 .

It is worth mentioning that the impulsive equation

$$\begin{cases}(t^2(t^2+1)x')'+4|x|^{1/2}x-10(t^2+1)|x|^{-1/2}x=\sin t, & t\neq i,\quad t\geq 1,\\ \Delta(t^2(t^2+1)x')+2i^2(i^2+1)|x|^{1/2}x-2(i^2+1)^2|x|^{-1/2}x=2i, & t=i,\quad i>1,\end{cases}\tag{44}$$

obtained from (36) by setting $\Delta x = 0$, and the nonimpulsive equation

$$(t^2(t^2+1)x')'+4|x|^{1/2}x-10(t^2+1)|x|^{-1/2}x=\sin t,\quad t\geq 1,\tag{45}$$

obtained by removing impulses in (36) are nonoscillatory. This is illustrated in Figure 2 and 3 , respectively.

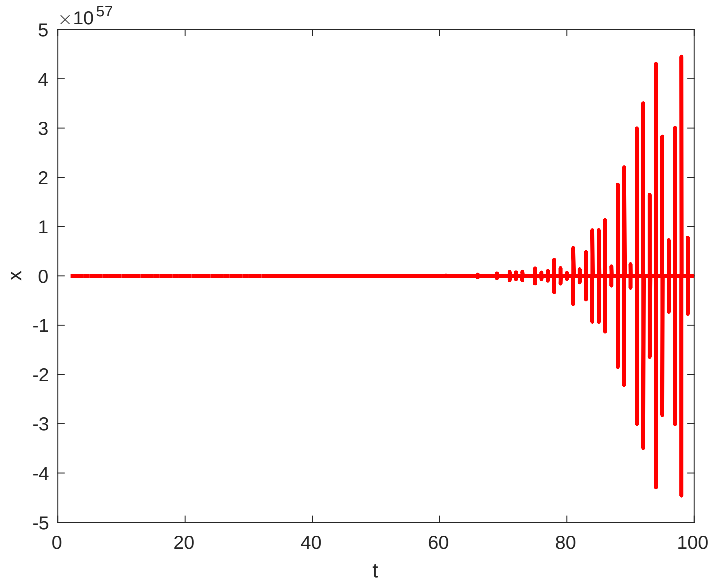

FIGURE 1 Graph of solution of (36) for $t \in [1, 100]$.

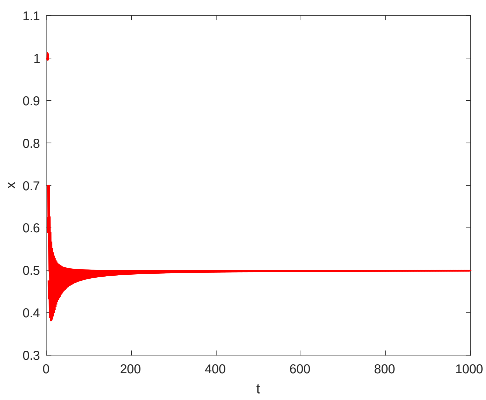

FIGURE 2 Graph of solution of (44)

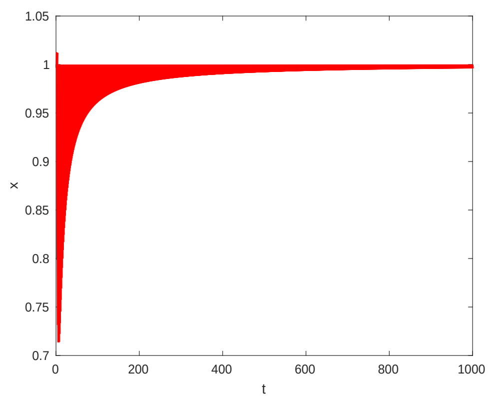

FIGURE 3 Graph of solution of (45)

Graphs of the solutions of equations (44) and (45) for $t \in [1, 1000]$.

References

1. J.S.W. Wong, Oscillation criteria for forced second-order linear differential equation, J. Math. Anal. Appl. 231 (1999) 235–240.
2. S. D. Akgöl, A. Zafer, Leighton and Wong type oscillation theorems for impulsive differential equations, Appl. Math. Lett. 121 (2021) 107513.
3. A. Özbekler, A. Zafer, Principal and nonprincipal solutions of impulsive differential equations with applications, Appl. Math. Comput., 216 (2010) 1158–1168.
4. A. Özbekler, J.S.W. Wong, A. Zafer, Forced oscillation of second-order nonlinear differential equations with positive and negative coefficients, Appl. Math. Lett. 24 (2011) 1225–1230.
5. A. Özbekler, A. Zafer, Forced oscillation of second-order impulsive differential equations with mixed nonlinearities, J. Differ. Equ. Appl., 47. Springer Proc. Math. Stat. New York: Springer; (2013) 183–195.
6. S. Doğru Akgöl, A. Zafer, Prescribed asymptotic behavior of second-order impulsive differential equations via principal and nonprincipal solutions, J. Math. Anal. Appl. 503 (2021) 125311.
7. W.T. Li, S.S. Cheng, An oscillation criterion for nonhomogeneous half-linear differential equations, Appl. Math. Lett. 15 (2002) 259–263.
8. Y.G. Sun, J.S.W. Wong, Oscillation criteria for second order forced ordinary differential equations with mixed nonlinearities, J. Math. Anal. Appl. 334 (2007), 549–560.
9. L. Xing, Z. Zheng, New oscillation criteria for forced second order halflinear differential equations with damping, Appl. Math.Comput. 198 (2008) 481—486.
10. A. Özbekler, A. Zafer, Wong's oscillation theorem for the second-order delay differential equations, J. Math. Sci. 222 (2017) 304—311.
11. X. Liu, Z. Xu, Oscillation criteria for a forced mixed type Emden–Fowler equation with impulses, Appl. Math. Comput. 215 (2009) 283–291.
12. X. Zhou, W.S. Wang, R. Chen, Interval oscillation criteria for forced mixed nonlinear impulsive differential equations with variable delay, Appl. Math. Lett. 75 (2018) 89–95.
13. X. Zhou, C. Liu, W.S. Wang, Interval oscillation criteria for nonlinear differential equations with impulses and variable delay, Appl. Math. Lett. 85 (2018) 150–156.
14. E. Thandapani, R. Sakthivel, E. Chandrasekaran, Oscillation of second order nonlinear impulsive differential equations with deviating arguments, Differ. Equ. Appl. 4 (2012) 571–580.
15. F. Karakoc, A. Ünal, H. Bereketoglu, Oscillation of a nonlinear impulsive differential equation system with piecewise constant argument, Adv. Differ. Equ. 99 (2018).

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