A generalized quasi-boundary value method for the backward nonlinear time-fractional diffusion problem in a cylinder

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Abstract

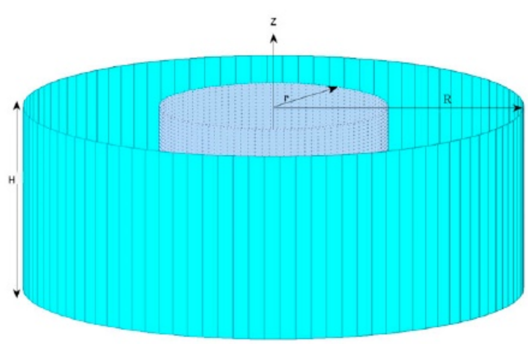
In this paper, a backward problem for a nonlinear time-fractional diffusion equation in an axis-symmetric cylinder has been considered. Under some assumptions, we prove the existence and uniqueness of the solution to the nonlinear problem. The ill-posedness of the backward problem is established and we obtain the error estimates by a generalized quasi-boundary value regularization method.

Keywords: time-fractional equation; backward problem; generalized quasi-boundary value regularization method; error estimates

1. Introduction

Nowadays, the diffusion equation is of great importance in science and engineering. It has a wide range of practical application such as aeronautics and astronautics, atomic energy technology, metallurgy, etc. The blast furnace steelmaking is an important technology in metallurgy, one can deduce the thickness of the wall by using the measured temperature outside, which can effectively avoid production accidents and ensure lower production costs (figure a).

(a) ***The blast furnace steelmaking***



(b) ***An axis-symmetric cylinder model***

Figure b shows an simplified blast furnace steelmaking model in an axis-symmetric cylinder which

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can be expressed mathematically as follows

$$u_t(r,z,t)=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}+f(r,z,t).$$

Although the classical integer order differential equation has acquired rather good effect on the practical production as mentioned above, time-fractional diffusion equation can be more accurate to describe the subdiffusion and superdiffusion phenomena in many areas [1–5]. We notice that the forward problems of the time-fractional diffusion equation has been investigated extensively, while the backward problems has not been studied sufficiently in spite of the significance. Actually, many phenomena cannot be observed at the time $t=0$, which means that the initial data may not be known. On the contrary, the phenomena can be measured at the backward time $t=T$ in some situations such as the temperature measurement in the blast furnace steelmaking. Recently, scholars have done a lot of research on the backward problems for the time-fractional diffusion equation [6–8].

We note that there are more one-dimensional works and less two-dimensional ones. In [9–12], the authors obtained the solution of backward problem of the one-dimensional time-fractional diffusion equation. In [13], the authors solved the backward problem of the two-dimensional nonlinear time-fractional diffusion equation.

In this article, we focus on the following two-dimensional nonlinear time-fractional diffusion equation in an axis-symmetric cylinder

$$D_t^{\alpha}u(r,z,t)=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}+f(r,z,t,u(r,z,t)), \tag{1.1}$$

subjected to the following initial and boundary conditions given respectively as

$$u(r,z,0)=h(r,z), \qquad 0\le r\le R, 0\le z\le L, \tag{1.2}$$

and

$$\lim_{r\to 0}u(r,z,t) \quad \textit{bounded}, \qquad 0\le z\le L, 0\le t\le T, \tag{1.3}$$

$$u(R,z,t)=0, \qquad 0\le z\le L, 0\le t\le T, \tag{1.4}$$

$$u(r,0,t)=u(r,L,t)=0, \qquad 0\le r\le R, 0\le t\le T, \tag{1.5}$$

where $D_t^{\alpha}u(r,z,t)$ is the Caputo fractional derivatives of order $0<\alpha<1$ [14], $f(r,z,t,u(r,z,t))$ is a nonlinear source term.

If all the data $f(r,z,t,u(r,z,t))$ and $h(r,z)$ are given, then problem (1.1)-(1.5) is a direct problem for the time fractional diffusion equation. From the information given at final time

$$u(r,z,T)=g(r,z), \qquad 0\le r\le R, 0\le z\le L, \tag{1.6}$$

the goal of the inverse problem is to recover the information $\{u(r,z,t), h(r,z)$ initial condition$\}$for $0 \le t < T$. Since the measurement is always noise-contaminated, thus we have only the measurement data $g^\delta(r,z)$ satisfying

$$\|g^\delta(r,z) - g(r,z)\| \le \delta, \tag{1.7}$$

where $\|\cdot\|$ is the $L^2_r([0,R]\times[0,L])$ norm and $\delta > 0$ indicates a noise level.

The remainder of this paper is organized as follows. In Section 2, we give the existence and uniqueness of the solution to the backward problem. The instablity of the solution is analyzed in Section 3. Moreover, we introduce a generalized quasi-boundary value regularization method and provide the error estimates.

2. Existence and uniqueness of the solution

We introduce the following Lebesgue space [6]

$$L^2_r(\Omega) = \{v : \Omega \to \mathbb{R} \quad measurable; \int_\Omega v^2(r,z) r dr dz < +\infty\},$$

which is a Hilbert space with the inner product

$$(u,v)_r = \int_\Omega u(r,z)v(r,z) r dr dz,$$

and the corresponding norm is defined by

$$\|v\|_{L^2_r(\Omega)} = (\int_\Omega v^2(r,z) r dr dz)^{\frac{1}{2}},$$

where $\Omega = ([0,R]\times[0,L])$.

Definition 2.1 [14]. The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Lemma 2.1 [14]. *Let $\alpha \in (0,1)$ then $E_{\alpha,1}(-z) > 0$ for any $z > 0$. Moreover, there exist three positive constants $\mathcal{M}^-_{1,\alpha}$, $\mathcal{M}^+_{1,\alpha}$, $\mathcal{M}^+_{2,\alpha}$ such that*

$$\frac{\mathcal{M}^-_{1,\alpha}}{1+z} \le E_{\alpha,1}(-z) \le \frac{\mathcal{M}^+_{1,\alpha}}{1+z}, \quad E_{\alpha,\alpha}(-z) \le \frac{\mathcal{M}^+_{2,\alpha}}{1+z}. \tag{2.1}$$

If $\alpha \in [\alpha_0, \alpha_1]$ for any $0 < \alpha_0 < \alpha_1 < 1$ then by [19],the constants $\mathcal{M}^-_{1,\alpha}$, $\mathcal{M}^+_{1,\alpha}$, $\mathcal{M}^+_{2,\alpha}$ can be chosen which depends only α_0, α_1.

Lemma 2.2 [15]. *For $\lambda > 0$ and $0 < \alpha < 1$,we have*

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0.$$

By the separation of variables and Laplace transform of Mittag-Leffler function, we can get the solution of problem (1.1)-(1.5) as follows

$$u(r,z,t)=\sum_{m,n=1}^{\infty}[h_{mn}E_{\alpha,1}(-\lambda_{mn}t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{mn}(t-\tau)^{\alpha})f_{mn}(u)(\tau)d\tau]\omega_{mn}(r,z), \tag{2.2}$$

where $h_{mn}=(h(r,z),\omega_{mn}(r,z))_r$, $f_{mn}(u)(\tau)=(f(r,z,\tau,u(r,z,\tau)),\omega_{mn}(r,z))_r$, $\lambda_{mn}=(\frac{\mu_n}{R})^2+(\frac{m\pi}{L})^2$. The eigenfunctions $\omega_{mn}(r,z)=\frac{2}{\sqrt{L}RJ_1(\mu_n)}J_0(\frac{\mu_n r}{R})sin(\frac{m\pi}{L}z)$ form an standard orthogonal basis in $L_r^2(\Omega)$ while the $J_0(x)$ and $J_1(x)$ denote the $0th$ order and $1st$ order Bessel functions, $\{\mu_n\}_{n=1}^{\infty}$ are the sequence of the zeros of $J_0(x)$.

Applying the final value data $u(r,z,T)=g(r,z)$, we can express the solution of the backward problem(1.1),(1.3)-(1.6) as follows

$$\begin{aligned}u(r,z,t)=&\sum_{m,n=1}^{\infty}\frac{E_{\alpha,1}(-\lambda_{mn}t^{\alpha})}{E_{\alpha,1}(-\lambda_{mn}T^{\alpha})}g_{mn}\omega_{mn}(r,z)\\&-\sum_{m,n=1}^{\infty}\int_0^T(T-\tau)^{\alpha-1}\frac{E_{\alpha,1}(-\lambda_{mn}t^{\alpha})E_{\alpha,\alpha}(-\lambda_{mn}(T-\tau)^{\alpha})}{E_{\alpha,1}(-\lambda_{mn}T^{\alpha})}f_{mn}(u)(\tau)d\tau\omega_{mn}(r,z)\\&+\sum_{m,n=1}^{\infty}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{mn}(t-\tau)^{\alpha})f_{mn}(u)(\tau)d\tau\omega_{mn}(r,z),\end{aligned} \tag{2.3}$$

where $g_{mn}=(g(r,z),\omega_{mn}(r,z))_r$.

Theorem 2.1. *Let the source term $f(r,z,t,u(r,z,t))$ satisfy*

$$\|f(u)(\cdot,\cdot,t))-f(v)(\cdot,\cdot,t)\|_{L_r^2(\Omega)}\le\mathcal{K}(t)\|u(\cdot,\cdot,t)-v(\cdot,\cdot,t)\|_{L_r^2(\Omega)} \tag{2.4}$$

for any $t\in[0,T]$, where $\mathcal{K}(t)>0$ is bounded, meanwhile the integral $\int_0^T(T-t)^{-2}\mathcal{K}^2(t)dt$ converges to $|\mathcal{P}(T)|^2$. Assume the following inequality holds

$$\mathcal{A}(\alpha,T)=T^{\frac{2\alpha+1}{2}}\mathcal{P}(T)\frac{\mathcal{M}_{2,\alpha}^{+}}{\mathcal{M}_{1,\alpha}^{-}}+\mathcal{Q}(\sup_{0\le t\le T}|\mathcal{K}(t)|)<1, \tag{2.5}$$

then problem (1.1),(1.3)-(1.6) has a unique solution in $L^{\infty}(0,T;L_r^2(\Omega))$.

Proof. Here we define the following operators

$$\mathcal{F}_1u(r,z,t)=-\sum_{m,n=1}^{\infty}\int_0^T(T-\tau)^{\alpha-1}\frac{E_{\alpha,1}(-\lambda_{mn}t^{\alpha})E_{\alpha,\alpha}(-\lambda_{mn}(T-\tau)^{\alpha})}{E_{\alpha,1}(-\lambda_{mn}T^{\alpha})}f_{mn}(u)(\tau)d\tau\omega_{mn}(r,z),$$

$$\mathcal{F}_2u(r,z,t)=\sum_{m,n=1}^{\infty}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{mn}(t-\tau)^{\alpha})f_{mn}(u)(\tau)d\tau\omega_{mn}(r,z).$$

From (2.4) and Lemma 2.1,we get

$$\|\mathcal{F}_1u(r,z,t)-\mathcal{F}_1v(r,z,t)\|_{L_r^2(\Omega)}^2$$

$$
\begin{aligned}
&= \sum_{m,n=1}^{\infty} \left(\int_0^T (T-\tau)^{\alpha-1} \frac{E_{\alpha,1}(-\lambda_{mn} t^\alpha) E_{\alpha,\alpha}(-\lambda_{mn}(T-\tau)^\alpha)}{E_{\alpha,1}(-\lambda_{mn} T^\alpha)} \left(f_{mn}(u)(\tau) - f_{mn}(v)(\tau)\right) d\tau \right)^2 \\
&\le T \left| \frac{\mathcal{M}^+_{2,\alpha}}{\mathcal{M}^-_{1,\alpha}} \right|^2 \sum_{m,n=1}^{\infty} \int_0^T (T-\tau)^{2\alpha-2} \frac{\lambda_{mn}^2 T^{2\alpha}}{\lambda_{mn}^2 (T-\tau)^{2\alpha}} \left(f_{mn}(u)(\tau) - f_{mn}(v)(\tau)\right)^2 d\tau \\
&\le T^{2\alpha+1} \left| \frac{\mathcal{M}^+_{2,\alpha}}{\mathcal{M}^-_{1,\alpha}} \right|^2 \int_0^T (T-\tau)^{-2} \left\| f(u)(\cdot,\cdot,\tau) - f(v)(\cdot,\cdot,\tau) \right\|^2_{L^2_r(\Omega)} d\tau \\
&\le T^{2\alpha+1} \left| \frac{\mathcal{M}^+_{2,\alpha}}{\mathcal{M}^-_{1,\alpha}} \right|^2 \int_0^T (T-\tau)^{-2} \mathcal{K}^2(\tau) \left\| u(\cdot,\cdot,\tau) - v(\cdot,\cdot,\tau) \right\|^2_{L^2_r(\Omega)} d\tau \\
&\le T^{2\alpha+1} \left| \frac{\mathcal{M}^+_{2,\alpha}}{\mathcal{M}^-_{1,\alpha}} \right|^2 |\mathcal{P}(T)|^2 \left\| u(\cdot,\cdot,\tau) - v(\cdot,\cdot,\tau) \right\|^2_{L^\infty(0,T;L^2_r(\Omega))}.
\end{aligned}
$$

Therefore

$$
\left\| \mathcal{F}_1 u(r,z,t) - \mathcal{F}_1 v(r,z,t) \right\|_{L^\infty(0,T;L^2_r(\Omega))} \le T^{\frac{2\alpha+1}{2}} \frac{\mathcal{M}^+_{2,\alpha}}{\mathcal{M}^-_{1,\alpha}} \mathcal{P}(T) \left\| u(\cdot,\cdot,\tau) - v(\cdot,\cdot,\tau) \right\|_{L^\infty(0,T;L^2_r(\Omega))}. \tag{2.6}
$$

From Lemma 2.2 together with $E_{\alpha,1}(0) = 1$, there holds

$$
\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_{mn}(t-\tau)^\alpha\right) d\tau = \frac{1}{\lambda_{mn}} (1 - E_{\alpha,1}(-\lambda_{mn} t^\alpha)) > 0.
$$

Thus

$$
\begin{aligned}
&\left\| \mathcal{F}_2 u(r,z,t) - \mathcal{F}_2 v(r,z,t) \right\|^2_{L^2_r(\Omega)} \\
&= \sum_{m,n=1}^{\infty} \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{mn}(t-\tau)^\alpha) \left(f_{mn}(u)(\tau) - f_{mn}(v)(\tau)\right) d\tau \right)^2 \\
&\le \left\| f(u)(\cdot,\cdot,\tau) - f(v)(\cdot,\cdot,\tau) \right\|^2_{L^\infty(0,T;L^2_r(\Omega))} \sum_{m,n=1}^{\infty} \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{mn}(t-\tau)^\alpha) d\tau \right)^2 \\
&\le \left(\sup_{0\le t\le T} |\mathcal{K}(t)| \right)^2 \left\| u(\cdot,\cdot,\tau) - v(\cdot,\cdot,\tau) \right\|^2_{L^\infty(0,T;L^2_r(\Omega))} \sum_{m,n=1}^{\infty} \frac{1}{\lambda_{mn}^2} \left(1 - E_{\alpha,1}(-\lambda_{mn} t^\alpha)\right)^2 \\
&\le \left(\sup_{0\le t\le T} |\mathcal{K}(t)| \right)^2 \left\| u(\cdot,\cdot,\tau) - v(\cdot,\cdot,\tau) \right\|^2_{L^\infty(0,T;L^2_r(\Omega))} \sum_{m,n=1}^{\infty} \frac{1}{\lambda_{mn}^2}.
\end{aligned}
$$

Noting that $\mathcal{Q}^2 = \sum_{m,n=1}^{\infty} \frac{1}{\lambda_{mn}^2} < \infty$, we obtain

$$
\left\| \mathcal{F}_2 u(r,z,t) - \mathcal{F}_2 v(r,z,t) \right\|_{L^\infty(0,T;L^2_r(\Omega))} \le \mathcal{Q} \sup_{0\le t\le T} |\mathcal{K}(t)| \left\| u(\cdot,\cdot,\tau) - v(\cdot,\cdot,\tau) \right\|_{L^\infty(0,T;L^2_r(\Omega))}. \tag{2.7}
$$

Define the following operator

$$
\mathcal{L}(u) = \sum_{m,n=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_{mn} t^\alpha)}{E_{\alpha,1}(-\lambda_{mn} T^\alpha)} g_{mn} \omega_{mn}(r,z) + \mathcal{F}_1 u(r,z,t) + \mathcal{F}_2 u(r,z,t).
$$

Combining (2.5), (2.6) and (2.7), we deduce that

$$\begin{aligned}\|\mathcal{L}(u)-\mathcal{L}(v)\|_{L^\infty(0,T;L^2_r(\Omega))} &\le \|\mathcal{F}_1 u-\mathcal{F}_1 v\|_{L^\infty(0,T;L^2_r(\Omega))}+\|\mathcal{F}_2 u-\mathcal{F}_2 v\|_{L^\infty(0,T;L^2_r(\Omega))}\\ &\le \mathcal{A}(\alpha,T)\,\|u-v\|_{L^\infty(0,T;L^2_r(\Omega))}. \end{aligned} \tag{2.8}$$

In view of (2.8) and noting that $\mathcal{A}(\alpha,T)<1$, we conlude that $\mathcal{L}$ is a contraction mapping. By using Banach fixed point theorem, we can deduce that problem (1.1),(1.3)-(1.6) has a unique solution in $L^\infty(0,T;L^2_r(\Omega))$.

3. Regularization and error estimate

In order to illustrate the ill-posedness of the backward problem (1.1),(1.3)-(1.6) through an example, we express the solution with noisy data as follows

$$u^\delta(r,z,t)=\sum_{m,n=1}^{\infty}\frac{E_{\alpha,1}(-\lambda_{mn}t^\alpha)}{E_{\alpha,1}(-\lambda_{mn}T^\alpha)}g^\delta_{mn}\omega_{mn}(r,z)+\mathcal{F}_1u^\delta(r,z,t)+\mathcal{F}_2u^\delta(r,z,t), \tag{3.1}$$

where $g^\delta=g+\frac{\omega_{ij}}{\sqrt{\lambda_{ij}}}$ (for some nature numbers i,j) is the possible measurement data. Substituting (2.8) into (3.1), we can get it immediately

$$\begin{aligned}&\|u^\delta(\cdot,\cdot,t)-u(\cdot,\cdot,t)\|_{L^2_r(\Omega)}\\ &\ge \left\|\sum_{m,n=1}^{\infty}\frac{E_{\alpha,1}(-\lambda_{mn}t^\alpha)}{E_{\alpha,1}(-\lambda_{mn}T^\alpha)}(g^\delta_{mn}-g_{mn})\omega_{mn}(r,z)\right\|_{L^2_r(\Omega)}-\mathcal{A}(\alpha,T)\left\|u^\delta-u\right\|_{L^\infty(0,T;L^2_r(\Omega))}\\ &=\frac{E_{\alpha,1}(-\lambda_{ij}t^\alpha)}{\sqrt{\lambda_{ij}}E_{\alpha,1}(-\lambda_{ij}T^\alpha)}-\mathcal{A}(\alpha,T)\left\|u^\delta-u\right\|_{L^\infty(0,T;L^2_r(\Omega))}.\end{aligned} \tag{3.2}$$

This implies that

$$\begin{aligned}&\|u^\delta-u\|_{L^\infty(0,T;L^2_r(\Omega))}+\mathcal{A}(\alpha,T)\left\|u^\delta-u\right\|_{L^\infty(0,T;L^2_r(\Omega))}=(1+\mathcal{A}(\alpha,T))\|u^\delta-u\|_{L^\infty(0,T;L^2_r(\Omega))}\\ &\ge \|u^\delta(\cdot,\cdot,t)-u(\cdot,\cdot,t)\|_{L^2_r(\Omega)}+\mathcal{A}(\alpha,T)\left\|u^\delta-u\right\|_{L^\infty(0,T;L^2_r(\Omega))}\\ &\ge \sup_{0\le t\le T}\frac{E_{\alpha,1}(-\lambda_{ij}t^\alpha)}{\sqrt{\lambda_{ij}}E_{\alpha,1}(-\lambda_{ij}T^\alpha)}\ge\frac{1}{\sqrt{\lambda_{ij}}E_{\alpha,1}(-\lambda_{ij}T^\alpha)}\ge\frac{1+\lambda_{ij}T^\alpha}{\sqrt{\lambda_{ij}}\mathcal{M}^+_{1,\alpha}}.\end{aligned}$$

Therefore,we derive that

$$\lim_{i,j\to\infty}\left\|u^\delta-u\right\|_{L^\infty(0,T;L^2_r(\Omega))}\ge\lim_{i,j\to\infty}\frac{1+\lambda_{ij}T^\alpha}{(1+\mathcal{A}(\alpha,T))\sqrt{\lambda_{ij}}\mathcal{M}^+_{1,\alpha}}=\infty, \tag{3.3}$$

which indicates that even if the noise level $\delta=\frac{1}{\sqrt{\lambda_{ij}}}$ goes to zero, as $i,j\to\infty$, the instability always happens in time. Hence, a regularization method is needed to restore the stability of the solution.

Here we provide a generalized quasi-boundary value regularization method which is inspired by the paper of [18]. In view of the complexity of nonlinear operator in two-dimensional cases, we define the regularized problem as follows

$$\begin{cases} D_t^\alpha u^{\delta,\nu}(r,z,t) = \dfrac{\partial^2 u^{\delta,\nu}}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u^{\delta,\nu}}{\partial r} + \dfrac{\partial^2 u^{\delta,\nu}}{\partial z^2} + f^{\delta,\nu}(r,z,t,u^{\delta,\nu}(r,z,t)), 0<r<R,\ 0<z<L, 0<t<T, \\ u^{\delta,\nu}(r,z,0) = h^{\delta,\nu}(r,z), & 0\le r\le R, 0\le z\le L, \\ \lim\limits_{r\to 0} u^{\delta,\nu}(r,z,t) \quad \textit{bounded}, & 0\le z\le L, 0\le t\le T, \\ u^{\delta,\nu}(R,z,t) = 0, & 0\le z\le L, 0\le t\le T, \\ u^{\delta,\nu}(r,0,t) = u^{\delta,\nu}(r,L,t) = 0, & 0\le r\le R, 0\le t\le T, \\ u^{\delta,\nu}(r,z,T) = g^\delta(r,z) - \nu(W)^\gamma h^{\delta,\nu}(r,z). & 0\le r\le R, 0\le t\le T, \end{cases} \tag{3.4}$$

where $\gamma \ge 0$ is a given real number, $\nu > 0$ is the regularization paramete, W is an operator satisfying $W^\gamma(h)(r,z) = \sum_{m,n=1}^\infty \lambda_{mn}^\gamma (h,\omega_{mn})\omega_{mn}(r,z)$, thus we can get the solution $u^{\delta,\nu}(r,z,t)$ of the regularized problem (3.4) as follows

$$\begin{aligned} u^{\delta,\nu}(r,z,t) = & \sum_{m,n=1}^\infty \frac{E_{\alpha,1}(-\lambda_{mn}t^\alpha)}{E_{\alpha,1}(-\lambda_{mn}T^\alpha) + \lambda_{mn}^\gamma \nu} g_{mn}^\delta \omega_{mn}(r,z) \\ & - \sum_{m,n=1}^\infty \int_0^T (T-\tau)^{\alpha-1} \frac{E_{\alpha,1}(-\lambda_{mn}t^\alpha)E_{\alpha,\alpha}(-\lambda_{mn}(T-\tau)^\alpha)}{E_{\alpha,1}(-\lambda_{mn}T^\alpha) + \lambda_{mn}^\gamma \nu} f_{mn}(u^{\delta,\nu})(\tau)d\tau\omega_{mn}(r,z) \\ & + \sum_{m,n=1}^\infty \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{mn}(t-\tau)^\alpha) f_{mn}(u^{\delta,\nu})(\tau)d\tau\omega_{mn}(r,z). \end{aligned} \tag{3.5}$$

Remark 3.1. We can compare the previous regularization method similarly in spite of the specific nonlinear operator in the quasi-boundary value condition. If we take $\gamma = 0$ in the above problem, the regularization method is parallel to the standard quasi-boundary value method, which is also the well known Lavrentiev regularization method for this case. The best convergence rate is $O(\delta^{\frac{1}{2}})$, see [16, 17] for solving backward problems of fractional diffusion equations. If taking $\gamma = 1$, it is analogous to the modified quasi-boundary value method, an optimal convergence order $O(\delta^{\frac{2}{3}})$ is obtained, see [18] for solving the inverse source problem of fractional diffusion equation.

Lemma 3.1 [19]. *For constants $\nu > 0$, $\beta > 0$, $\gamma \ge 0$, $-1 < \theta \le \gamma$, $s \ge \lambda_{11} > 0$, we have*

$$F(s) = \frac{s^{1+\theta}}{\nu s^{\gamma+1} + \beta} \le C_1 \nu^{-\frac{\theta+1}{\gamma+1}},$$

where $C_1 = C_1(\beta,\gamma,\theta) > 0$ is a constant independent of s.

Moreover for constants $p > 0$, $\nu > 0$, $\beta > 0$, $\gamma \ge 0$, $s \ge \lambda_{11} > 0$, then we have

$$G(s) = \frac{\nu s^{\gamma+1-\frac{p}{2}}}{\nu s^{\gamma+1} + \beta} \le \begin{cases} C_2 \nu^{\frac{p}{2(\gamma+1)}}, 0 < p < 2(\gamma+1), \\ C_3 \nu, p \ge 2(\gamma+1), \end{cases}$$

where $C_2 = C_2(p, \beta, \gamma) > 0$, $C_3 = C_3(p, \beta, \gamma, \lambda_{11}) > 0$ are constants independent of s.

Theorem 3.1. *Assume that there exists a priori bound condition such that*

$$\|g(r,z)\|_{\frac{p+2}{2}} \le E, \qquad p > 0, \tag{3.6}$$

where

$$\|g(r,z)\|_{\frac{p+2}{2}} = \left\| \sum_{m,n=1}^{\infty} (\lambda_{mn})^{\frac{p+2}{2}} g_{mn}\omega_{mn}(r,z) \right\|_{L_r^2(\Omega)} = \left(\sum_{m,n=1}^{\infty} (\lambda_{mn})^{p+2} g_{mn}^2 \right)^{\frac{1}{2}}.$$

Suppose the noise assumption (1.7) *holds, then we have*

(1) *If $0 < p < 2(\gamma+1)$, we have a convergence rate*

$$\|u^{\delta,\nu} - u\|_{L^\infty(0,T;L_r^2(\Omega))} \le C_4 E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}};$$

(2) *If $p \ge 2(\gamma+1)$, we have a convergence rate*

$$\|u^{\delta,\nu} - u\|_{L^\infty(0,T;L_r^2(\Omega))} \le C_5 E^{\frac{1}{\gamma+2}} \delta^{\frac{\gamma+1}{\gamma+2}},$$

where C_4, C_5 are positive constants depending on $p, \gamma, \lambda_{11}, \alpha, T, \mathcal{M}_{1,\alpha}^-$.

Proof. By the definition of $\mathcal{F}_1$ and $\mathcal{F}_2$, we obtain

$$\begin{aligned}\|u^{\delta,\nu} - u\|_{L_r^2(\Omega)} \le & \underbrace{\left\| \sum_{m,n=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_{mn}t^\alpha)}{E_{\alpha,1}(-\lambda_{mn}T^\alpha) + \lambda_{mn}^\gamma \nu} g_{mn}^\delta \omega_{mn}(r,z) - \sum_{m,n=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_{mn}t^\alpha)}{E_{\alpha,1}(-\lambda_{mn}T^\alpha)} g_{mn}\omega_{mn}(r,z) \right\|_{L_r^2(\Omega)}}_{\mathcal{D}_1} \\ & + \|\mathcal{F}_1 u^{\delta,\nu}(r,z,t) - \mathcal{F}_1 u(r,z,t)\|_{L_r^2(\Omega)} + \|\mathcal{F}_2 u^{\delta,\nu}(r,z,t) - \mathcal{F}_2 u(r,z,t)\|_{L_r^2(\Omega)}. \end{aligned} \tag{3.7}$$

Applying Lemma 2.1, we estimate $\mathcal{D}_1$ as follows

$$\begin{aligned}\mathcal{D}_1 \le & \left\| \sum_{m,n=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_{mn}t^\alpha)}{E_{\alpha,1}(-\lambda_{mn}T^\alpha) + \lambda_{mn}^\gamma \nu} (g_{mn}^\delta - g_{mn})\omega_{mn}(r,z) \right\|_{L_r^2(\Omega)} \\ & + \left\| \sum_{m,n=1}^{\infty} \frac{-\lambda_{mn}^\gamma \nu E_{\alpha,1}(-\lambda_{mn}t^\alpha)}{E_{\alpha,1}(-\lambda_{mn}T^\alpha)(E_{\alpha,1}(-\lambda_{mn}T^\alpha) + \lambda_{mn}^\gamma \nu)} g_{mn}\omega_{mn}(r,z) \right\|_{L_r^2(\Omega)} \\ \le & \left\| \sum_{m,n=1}^{\infty} \frac{\lambda_{mn}}{\underline{C} + \lambda_{mn}^{\gamma+1}\nu} (g_{mn}^\delta - g_{mn})\omega_{mn}(r,z) \right\|_{L_r^2(\Omega)} \\ & + \left\| \sum_{m,n=1}^{\infty} \left(\frac{-\lambda_{mn}^\gamma \nu}{E_{\alpha,1}(-\lambda_{mn}T^\alpha) + \lambda_{mn}^\gamma \nu)} \right) \frac{1}{\lambda_{mn}^{\frac{p}{2}}} \lambda_{mn}^{\frac{p}{2}} \left(\frac{E_{\alpha,1}(-\lambda_{mn}t^\alpha)}{E_{\alpha,1}(-\lambda_{mn}T^\alpha)} \right) g_{mn}\omega_{mn}(r,z) \right\|_{L_r^2(\Omega)}. \end{aligned} \tag{3.8}$$

Substituting the priori bound condition (3.6) and Lemma 3.1 into (3.8), we obtain

$$\mathcal{D}_1 \le \frac{C_1}{\nu^{\frac{1}{\gamma+1}}}\delta + \left(\sum_{m,n=1}^{\infty} \left(\frac{\lambda_{mn}^{\gamma+1-\frac{p}{2}}\nu}{\underline{C} + \lambda_{mn}^{\gamma+1}\nu} \right)^2 \underline{C}^2 (\lambda_{mn})^{p+2} g_{mn}^2 \right)^{\frac{1}{2}}$$

$$\leq \frac{C_1}{\nu^{\frac{1}{\gamma+1}}}\delta + E\begin{cases} C_2\nu^{\frac{p}{2(\gamma+1)}}, 0 < p < 2(\gamma+1), \\ C_3\nu, p \geq 2(\gamma+1), \end{cases} \tag{3.9}$$

where $\underline{C} = \frac{\mathcal{M}_{1,\alpha}^{-}}{1+T^{\alpha}}$ is a positive constant, C_1, C_2, C_3 are constants defined by Lemma 3.1.Combining (2.9), (3.7) and (3.9), we can easily obtain

$$\|u^{\delta,\nu} - u\|_{L_r^2(\Omega)} \leq \frac{C_1}{\nu^{\frac{1}{\gamma+1}}}\delta + E\begin{cases} C_2\nu^{\frac{p}{2(\gamma+1)}}, 0 < p < 2(\gamma+1), \\ C_3\nu, p \geq 2(\gamma+1), \end{cases} + \mathcal{A}(\alpha,T)\,\|u - v\|_{L^\infty(0,T;L_r^2(\Omega))}. \tag{3.10}$$

The right hand side of (3.10) is independent of t which implies that

$$\|u^{\delta,\nu} - u\|_{L^\infty(0,T;L_r^2(\Omega))} \leq \frac{C_1}{\nu^{\frac{1}{\gamma+1}}}\delta + E\begin{cases} C_2\nu^{\frac{p}{2(\gamma+1)}}, 0 < p < 2(\gamma+1), \\ C_3\nu, p \geq 2(\gamma+1), \end{cases} + \mathcal{A}(\alpha,T)\,\|u - v\|_{L^\infty(0,T;L_r^2(\Omega))}.$$

Since $\mathcal{A}(\alpha,T) < 1$, we obtain

$$\|u^{\delta,\nu} - u\|_{L^\infty(0,T;L_r^2(\Omega))} \leq \frac{C_1}{(1-\mathcal{A}(\alpha,T))\nu^{\frac{1}{\gamma+1}}}\delta + \frac{E}{1-\mathcal{A}(\alpha,T)}\begin{cases} C_2\nu^{\frac{p}{2(\gamma+1)}}, 0 < p < 2(\gamma+1), \\ C_3\nu, p \geq 2(\gamma+1). \end{cases} \tag{3.11}$$

Choosing the regularization parameter ν by

$$\nu = \begin{cases} \left(\dfrac{\delta}{E}\right)^{\frac{2(\gamma+1)}{p+2}}, 0 < p < 2(\gamma+1), \\ \left(\dfrac{\delta}{E}\right)^{\frac{\gamma+1}{\gamma+2}}, p \geq 2(\gamma+1), \end{cases}$$

then we deduce that

$$\|u^{\delta,\nu} - u\|_{L^\infty(0,T;L_r^2(\Omega))} \leq \begin{cases} \dfrac{C_1+C_2}{1-\mathcal{A}(\alpha,T)}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, 0 < p < 2(\gamma+1), \\ \dfrac{C_1+C_3}{1-\mathcal{A}(\alpha,T)}E^{\frac{1}{\gamma+2}}\delta^{\frac{\gamma+1}{\gamma+2}}, p \geq 2(\gamma+1). \end{cases}$$

The proof is completed.

Remark 3.2. We note that the convergence rates of the generalized quasi-boundary value method have no saturation phenomena. If we choose an appropriate γ such that the convergence rate $O(\delta^{\frac{\gamma+1}{\gamma+2}})$ can be better than $O(\delta^{\frac{2}{3}})$ for the fixed $\gamma = 1$ or $O(\delta^{\frac{1}{2}})$ for the fixed $\gamma = 0$.

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