BEURLING'S THEOREM FOR THE CLIFFORD-FOURIER TRANSFORM

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ABSTRACT. We provide a generalization of Beurling's theorem for the Clifford-Fourier transform and we give some of its applications. Indeed, analogues of Hardy, Cowling-Price and Gelfand-Shilov theorems are obtained in the Clifford analysis setting.

Keywords: Clifford analysis; Clifford-Fourier transform; Uncertainty principles; Beurling's theorem.

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1. INTRODUCTION

Uncertainty principle asserts that a function and its Fourier transform cannot both be sharply localized. In Euclidien spaces, many theorems are devoted to clarify it such as Beurling, Cowling and Price, Hardy, Heisenberg.
Beurling's theorem which is given by A. Beurling [1] and proved by Hörmander [9] is the the most relevant one: that is it gives Hardy, Cowling-Price and Gelfand-Shilov theroems. The result of Beurling-Hörmander describes the uncertainty principle in terms of a single integral estimate of f and its Fourier transform $\hat{f}$ as follows.

Theorem 1.1. [9] *Let $f \in L^2(\mathbb{R})$ be such that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)||\widehat{f}(y)| e^{|x||y|} dxdy < \infty,$$

where the Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(y) = \int_{\mathbb{R}} e^{-ixy} f(x) dx.$$

Then $f = 0$ almost everywhere.

This theorem is generalized by Bonami et al [2] by giving solutions in terms of Hermite functions.

Theorem 1.2. [2, Theorem 1.1] *Let $N \geq 0$. Assume that $f \in L^2(\mathbb{R}^m)$. Then, f satisfy*

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{|f(x)||\widehat{f}(y)|}{(1 + |x| + |y|)^N} e^{|x||y|} dxdy < \infty,$$

if and only if $f(x) = P(x)e^{-a|x|^2}$ *where* P *is a polynomial of degree* $< \frac{N-m}{2}$ *and* $a > 0$.

For $N = 0$, we should note that the hypothesis of Theorem 1.2 is the same as the one of Theorem 1.1. This case was studied by Bonami et al in [2, Appendix, p52] and it was proved that f is identically zero.
For $N \leq m$, we can write f as $f(x) = P(x)e^{-a|x|^2}$ using the result of Theorem 1.2, where P is a polynomial of negative degree. This yields that $P = 0$. Therefore, $f = 0$.
The Beurling's theorem was obtained for many Fourier transforms [11, 13]. In 2010, Kawazoe and Majjaoli provided a Beurling-type theorem for the Dunkl transform [10]. Moreover, Parui and Pusti gave an alternative proof of the Beurling's theorem for the Dunkl transform [12] similar to the proof used in [2].
Then, many studies were devoted to generalize Fourier transform in the Clifford analysis setting which is characterized by a lack of commutativity. In 2005, Sommen et al defined a generalization of the classical Fourier transform in the Clifford analysis setting called The Clifford-Fourier transform [3]. This transform was studied in [4, 6]. Following that, there was an interest in studying uncertainty principles in the Clifford analysis. In this context, many uncertainty principles for the Clifford-Fourier transform were proved, such as the Hardy theorem and the Heisenberg inequality [5, 7, 8]. However, the Cowling and Price theorem is not yet shown in the Clifford analysis setting. Furthermore, the Beurling's theorem, which is the most relevant one since it yields various other uncertainty principle theorems, is not provided.
Our goal in this paper is to establish the Beurling's theorem in the Clifford analysis setting. Indeed, we aim to prove Theorem 1.2 for the Clifford-Fourier transform defined by Sommen et al in [3].
This paper is organized as follows. In section 2, we recall the Clifford algebra and some notations that will be useful in the sequel. In section 3, we recall some aspects of the Clifford-Fourier transform and its properties. In section 4, we prove a Beurling-type theorem for the Clifford-Fourier transform. Section 5 contains some consequences of the Beurling's theorem for the Clifford-Fourier transform. Indeed, we derive the Hardy, Cowling and Price and Gelfand Shilov uncertainty principles in Clifford analysis.

2. Notations and preliminaries

The real Clifford algebra $\mathcal{C}l_{0,m}$ is generated by $\{1, e_1, \cdots, e_m\}$ with the multiplication rules

$$\begin{cases} e_i e_k = -e_k e_i, & \text{if} \quad i \neq k, \\ e_i^2 = -1, & \forall 1 \leq i \leq m. \end{cases} \tag{2.1}$$

We recall that $\mathcal{C}l_{0,m}$ can be decomposed as

$$\mathcal{C}l_{0,m} = \oplus_{k=0}^{m} \mathcal{C}l_{0,m}^{k}, \tag{2.2}$$

where $\mathcal{C}l_{0,m}^{k} = span\{e_{i_1} \cdots e_{i_k},\, i_1 < \cdots < i_k\}$.
We denote by $\mathcal{C}_m = \mathbb{C} \otimes \mathcal{C}l_{0,m}$ the complexification of $\mathcal{C}l_{0,m}$ which can be seen as $\mathcal{C}_m = \mathcal{C}l_{0,m} \oplus j\mathcal{C}l_{0,m}$ where j is the complex imaginary unit.

The set $\{e_A : A \subseteq \{1, \cdots, m\}\}$, with $e_A = e_{i_1} e_{i_2} \cdots e_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq m$, $e_\emptyset = 1$, forms a graded basis of $\mathcal{C}l_{0,m}$ and of $\mathcal{C}_m$.
A multivector x in the Clifford algebra $\mathcal{C}l_{0,m}$ (respectively $\mathcal{C}_m$) can be presented as:

$$x = \sum_{A \in L} e_A x_A, \tag{2.3}$$

where $L := \{i_1, \cdots, i_k\} \subset \{1, \cdots, m\}$, $e_A = e_{i_1} e_{i_2} \cdots e_{i_k}$ are ordered by $1 \leq i_1 < \cdots < i_k \leq m$ and x_A are real numbers(respectively complex numbers).
Conjugation in $\mathcal{C}l_{0,m}$ is defined as the anti-involution for which $\overline{e_k} = -e_k$, $k = 1, 2, ..., m$. In the case of $\mathcal{C}_m$, we add the rule $\overline{j} = -j$.
If x is an arbitrary element of $\mathcal{C}_m$, then its norm $||x||_c$ is:

$$||x||_c^2 = [\overline{x}x]_0 = \sum_{A \in L} |x_A|^2, \tag{2.4}$$

where $[\cdot]_0$ denotes the scalar part of the expression between brackets.
An element $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$ can be identified with the vector $x = \sum_{i=1}^{m} e_i x_i$.
The mutiplication of two vectors x and y is given by

$$xy = - < x, y > + x \wedge y, \tag{2.5}$$

where the inner product and the wedge product are defined respectively by

$$< x, y > = \sum_{k=1}^{m} x_k y_k = \frac{-1}{2}(xy + yx) \tag{2.6}$$

and

$$x \wedge y = \sum_{i<k} e_i e_k (x_i y_k - x_k y_i) = \frac{1}{2}(xy - yx). \tag{2.7}$$

We introduce the Dirac operator, Gamma operator and Laplace operator associated to a vector x respectively by:

$$\partial_x = \sum_{i=1}^{m} e_i \partial_{x_i}; \tag{2.8}$$

$$\Gamma_x = -\sum_{i<k} e_i e_k (x_i \partial_{x_k} - x_k \partial_{x_i}); \tag{2.9}$$

$$\Delta_x = \sum_{i=1}^{m} \partial_{x_i}^2. \tag{2.10}$$

In $\mathcal{C}l_{0,m}$, for a vector x, we have the following relations:

$$||x||_c^2 = -x^2 = \sum_{i=1}^{m} x_i^2 \tag{2.11}$$

and

$$\Delta_x = -\partial_x^2. \tag{2.12}$$

In the sequel, we consider functions defined on $\mathbb{R}^m$ and taking values in $\mathcal{C}l_{0,m}$ or its complexification $\mathcal{C}_m$. These functions can be written as :

$$f(x) = f_0(x) + \sum_{i=1}^{m} e_i f_i(x) + \sum_{i<k} e_i e_k f_{ik}(x) + \cdots + e_{1\cdots m} f_{1\cdots m}(x), \tag{2.13}$$

where $f_0, f_i, \cdots, f_{1\cdots m}$ all real-valued or complex- valued functions depending on whether the function takes values in $\mathcal{C}l_{0,m}$ or $\mathcal{C}_m$.
We denote by $\mathcal{P}_k$ the space of homogeneous polynomials of degree k taking values in $\mathcal{C}l_{0,m}$. $\mathcal{P}$ denotes the space of polynomials taking values in $\mathcal{C}l_{0,m}$. The right Hilbert module denoted by $L^2(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$ is the right $\mathcal{C}l_{0,m}$-module of square integrable functions taking values in $\mathcal{C}l_{0,m}$ equipped with the inner product

$$< f, g >= \int_{\mathbb{R}^m} \overline{f(x)} g(x) dx, \qquad \forall f, g \in L^2(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$$

and the associated norm

$$||f||_{2,c}^2 = [< f, f >]_0.$$

Finally, by respectively, $L^1(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$ and $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$, we denote the right $\mathcal{C}l_{0,m}$-modules of $\mathcal{C}l_{0,m}$-valued respectively integrable and rapidly decreasing functions.

3. Clifford-Fourier Transform

We introduce the class of functions
(3.1)

$$B(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m} = \left\{ f \in L^1(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m} / ||f||_B := \int_{\mathbb{R}^m} (1 + ||y||_c)^{\frac{m-2}{2}} ||f(y)||_c dy < \infty \right\}$$

Definition 3.1. [6] *The Clifford-Fourier transform is defined on $B(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$ by*

$$\mathcal{F}_\pm(f)(y) = (2\pi)^{\frac{-m}{2}} \int_{\mathbb{R}^m} K_\pm(x, y) f(x) dx, \tag{3.2}$$

where

$$K_\pm(x, y) = e^{\mp i \frac{\pi}{2} \Gamma_y} e^{-i<x,y>}. \tag{3.3}$$

We recall the characterization of the kernel in the following lemma which plays an important role in the main theorem. More precisely, it allows to determine the boundedness of the functions used in the proof.

Lemma 3.1. [8, Lemma 3.2] *Let m be even. Then*

$$||K_\pm(x, y)||_c \leq C e^{||x||_c ||y||_c}, \qquad \forall x, y \in \mathbb{R}^m. \tag{3.4}$$

The following theorem gives boundedness of the Clifford-Fourier transform.

Theorem 3.2. *Let m be even and $f \in B(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$. Then, there exists a positive constant A such that*

$$||\mathcal{F}_{\pm}(f)(y)||_c \leq Ce^{\frac{||y||_c^2}{4}}||f||_B, \quad \forall ||y||_c > A \tag{3.5}$$

and

$$||\mathcal{F}_{\pm}(f)(y)||_c \leq C(1+A)^{\frac{m-2}{2}}||f||_B, \quad \forall ||y||_c \leq A. \tag{3.6}$$

Proof. Recall that the Clifford kernel [6, Theorem 5.3] is written as:

$$K_-(x,y) = K_0^-(x,y) + \sum_{i<j} e_{ij}K_{ij}^-(x,y),$$

where $K_0^-(x,y)$ and $K_{ij}^-(x,y)$ satisfy

$$|K_0^-(x,y)| \leq C(1+||x||_c)^{\frac{m-2}{2}}(1+||y||_c)^{\frac{m-2}{2}}$$
$$|K_{ij}^-(x,y)| \leq C(1+||x||_c)^{\frac{m-2}{2}}(1+||y||_c)^{\frac{m-2}{2}}.$$

Thus,

$$||\mathcal{F}_{\pm}(f)(y)||_c \leq C(1+||y||_c)^{\frac{m-2}{2}}||f||_B.$$

Moreover, there exists $A > 0$ such that for all $||y||_c > A$,

$$(1+||y||_c)^{\frac{m-2}{2}} \leq Ce^{\frac{||y||_c^2}{4}},$$

which completes the proof. ■

We should return to the inversion theorem and the Plancherel theorem for the Clifford-Fourier transform.

Theorem 3.3. [4]
1) The Clifford-Fourier transform is a continuous operator mapping from $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$ to $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$ (see [4, Theorem 6.3]).
In particular, when m is even, we have

$$\mathcal{F}_{\pm}\mathcal{F}_{\pm} = id_{\mathcal{S}(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}}.$$

2) The Clifford-Fourier transform extends from $\mathcal{S}(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$ to a continuous map on $L^2(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$ (see [4, Theorem 6.4]).
In particular, when m is even, we have

$$||\mathcal{F}_{\pm}(f)||_{2,c} = ||f||_{2,c},$$

for all $f \in L^2(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$.

We now give the Clifford-Fourier transform for a function defined by the product of a polynomial and a Gaussian function. This function appears in Beurling's theorem and in other uncertainty principles. Specifically, in order to find that the function f is written in this form, it is enough to express the Clifford-Fourier transform in this form and to apply the inversion theorem and this theorem.

Theorem 3.4. [8, Theorem 3.5] *Let $a>0$ and $P\in\mathcal{P}_k$. Then, there exists $Q\in\mathcal{P}_k$ satisfying :*

$$\mathcal{F}_{\pm}(P(.)e^{-a||\cdot||_c^2})(x)=Q(x)e^{-\frac{||x||_c^2}{4a}}. \tag{3.7}$$

We recall the Clifford translation and the Clifford convolution, which generalize the classical translation and convolution.

Definition 3.2. [6] *Let m be even. The Clifford translation and the Clifford convolution for $f,g\in\mathcal{S}(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$ are introduced respectively by*

$$T_yf(x)=(2\pi)^{-\frac{m}{2}}\int_{\mathbb{R}^m}\overline{K_-(\epsilon,x)}K_-(y,\epsilon)\mathcal{F}(f)(\epsilon)d\epsilon, \tag{3.8}$$

$$f*_{Cl}g(x)=(2\pi)^{-\frac{m}{2}}\int_{\mathbb{R}^m}T_yf(x)g(y)dy. \tag{3.9}$$

The following theorem explicitly gives the Clifford translation for $m=2$ and for radial function.

Theorem 3.5. [6, Proposition 7.2, Theorem 7.3] *Let $f\in\mathcal{S}(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$.*
i) For $m=2$,

$$T_yf(x)=f(x-y).$$

ii) For m even and $m>2$, we have

$$T_yf(x)=f_0(|x-y|),$$

for a radial function f on $\mathbb{R}^m$, $f(x)=f_0(|x|)$ with $f_0:\mathbb{R}_+\to\mathbb{R}$.

In the next theorem, the action of the Clifford-Fourier Transform on Clifford's convolution is analogous to the classic case, but confined to radial functions.

Theorem 3.6. [6, Theorem 8.2] *Let $f\in\mathcal{S}(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$ be a radial function and $g\in\mathcal{S}(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$. Then,*

$$\mathcal{F}_{\pm}(f*_{Cl}g)=\mathcal{F}_{\pm}(f)\mathcal{F}_{\pm}(g).$$

In particular, we have

$$f*_{Cl}g=g*_{Cl}f.$$

In the following section, we will use the inversion theorem for functions belonging to $L^2(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$. Therefore, we need results associated with Lebesgue spaces. In this regard, we remember the next theorem of Young inequalities.

Theorem 3.7. *Let m be even integer. Let $1\le p,q<\infty$ and $r\ge 1$ with $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$.*

(1) [7, Theorem 5.2]
For $m=2$, if $f\in L^p(\mathbb{R}^2)\otimes\mathcal{C}l_{0,2}$ and $g\in L^q(\mathbb{R}^2)\otimes\mathcal{C}l_{0,2}$, then $f*_{Cl}g\in L^r(\mathbb{R}^2)\otimes\mathcal{C}l_{0,2}$.
(2) [7, Theorem 5.4]
For $m>2$, if $f(x)=f_0(||x||_c)$ is a real-valued radial function in $L^p(\mathbb{R}^m)$ and $g\in L^q(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$. Then $f*_{Cl}g\in L^r(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$.

In the next section, we will define the Clifford-Fourier transform for certain functions in the proof of Beurling's Theorem. Because the Clifford-Fourier transform is defined on $B(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$, it is critical to check whether or not these functions belong to $B(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$. The following corollary guarentees that.

Corollary 3.8. [7, Corollary 5.5] *Let $m>2$ be even. Suppose that $f(x)=f_0(||x||_c)$ is a real-valued radial function in $B(R^m)$ and $g\in B(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$. Then $f *_{Cl} g\in B(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$*

4. Beurling's theorem for the Clifford-Fourier transform

In this section, we provide Beurling's theorem for the Clifford-Fourier transform. The proof is based on considering a function g which is a Clifford convolution product of a Clifford-valued function f and a Gaussian function. In the beginning, we prove that the function $g(z)g(jz)$ is a polynomial. Indeed , we show that the function Γ defined below is an entire function bounded by a polynomial, so a polynomial. Then, by differentiating, it follows that $g(z)g(jz)$ is a polynomial. The main problem is proving that Γ is polynomial growth. For that purpose, we shall use the following theorem of Phragmen-Lindelholf.

Theorem 4.1. [14] *Let ϕ be an entire function of order 2 in the complex plane and let $\alpha\in]0,\frac{\pi}{2}[$. Assume that $|\phi(z)|$ is bounded by $C(1+|z|)^N$ on the boundary of some angular sector $\{re^{j\beta}: r\geq 0, \beta_0\leq\beta\leq\beta_0+\alpha\}$. Then the same bound is valid inside the angular sector* (when replacing C by $2^N C$).

Afterwards, we resolve the equation $g(z)g(jz)=R(z)$ where R is a polynomial. A next lemma application is used to write g as $g(x)=P(x)e^{-a||x||_c^2}$.

Lemma 4.2. [2] *Let ϕ be an entire function of order 2 on $\mathbb{C}^m$ such that, on every complex line, either ϕ is identically 0 or it has at most N zeros. Then, there exists a polynomial P with degree at most N and a polynomial Q with degree at most 2 such that $\phi(z)=P(z)e^{Q(z)}$.*

Because the Clifford-Fourier transform is defined on $B(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$, we need check using the hypothesis of the Beurling's theorem, if f belongs to $B(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$.

Lemma 4.3. *Assume $f\in L^2(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$ satisfying*

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{||f(x)||_c||\mathcal{F}_\pm(f)(y)||_c}{(1+||x||_c+||y||_c)^N}e^{||x||_c||y||_c}dxdy<\infty, \tag{4.1}$$

for some $N\geq 0$.
Then, $f\in B(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$ and $\mathcal{F}_\pm(f)\in B(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$.

Proof. We suppose that $f\neq 0$.
Applying Fubini's theorem, we obtain for almost every $y\in\mathbb{R}^m$,

$$||\mathcal{F}_\pm(f)(y)||_c\int_{\mathbb{R}^m}\frac{||f(x)||_c}{(1+||x||_c+||y||)^N}e^{||x||_c||y||_c}dx<\infty.$$

Since $f \neq 0$, then $\mathcal{F}_{\pm}(f) \neq 0$. Thus, there exists $y_0 \neq 0$ such that $\mathcal{F}_{\pm}(f)(y_0) \neq 0$ and

$$\int_{\mathbb{R}^m} \frac{||f(x)||_c}{(1+||x||_c)^N} e^{||x||_c||y_0||_c} dx < \infty. \tag{4.2}$$

Using (4.2) and the fact that for large x

$$\frac{e^{||x||_c||y_0||_c}}{(1+||x||_c)^{N+\frac{m-2}{2}}} \geq 1,$$

it follows that

$$\int_{\mathbb{R}^m} (1+||x||_c)^{\frac{m-2}{2}} ||f(x)||_c dx < \infty.$$

Hence, we find that $f \in B(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$. Similarly, we get $\mathcal{F}_{\pm}(f) \in B(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$. ■

Now, we state the main result which is the Beurling's theorem.

Theorem 4.4. *Let m be even and N be a positive integer. Assume $f \in L^2(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$ such that*

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \frac{||f(x)||_c ||\mathcal{F}_{\pm}(f)(y)||_c}{(1+||x||_c+||y||_c)^N} e^{||x||_c||y||_c} dxdy < \infty. \tag{4.3}$$

Then,

$$f(x) = Q(x) e^{-a||x||_c^2}.$$

for some $a > 0$ and polynomial Q with degree less than $\dfrac{N-m}{2}$.

Proof. Let

$$g(x) = f *_{Cl} e^{-\frac{||\cdot||_c^2}{2}}(x).$$

Step 1.
In this step, we want to establish (4.8). To do this, we need to prove few properties. Because the inequality (4.8) contains the Clifford-Fourier transform of g, we begin with check that g belongs to $B(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$.
By Lemma 4.3, we have $f \in B(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$. In order to demonstrate that $g \in B(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$, we distinguish two cases depending on m.
Let $m = 2$.
We notice that $B(\mathbb{R}^2) \otimes \mathcal{C}l_{0,2}$ is equivalent to $L^1(\mathbb{R}^2) \otimes \mathcal{C}l_{0,2}$. Thus, since $f \in B(\mathbb{R}^2) \otimes \mathcal{C}l_{0,2}$, an application of Theorem 3.7 gives that g belongs to $B(\mathbb{R}^2) \otimes \mathcal{C}l_{0,2}$.
Let $m > 2$.
According to Corollary 3.8, we have $g \in B(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$.
We conclude for m even, $g \in B(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$.
Theorem 3.6 and Theorem 3.4 yield

$$\mathcal{F}_{\pm}(g)(x) = \mathcal{F}_{\pm}(f)(x)\mathcal{F}_{\pm}(e^{-\frac{||\cdot||_c^2}{2}})(x) = \mathcal{F}_{\pm}(f)(x) e^{-\frac{||x||_c^2}{2}}. \tag{4.4}$$

We will show that g satisfies the following assumptions :

$$\int_{\mathbb{R}^m} ||\mathcal{F}_{\pm}(g)(y)||_c e^{\frac{||y||_c^2}{2}} dy < \infty, \tag{4.5}$$

$$||\mathcal{F}_{\pm}(g)(y)||_c \leq Ce^{-\frac{||y||_c^2}{4}}, \tag{4.6}$$

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{||g(x)||_c||\mathcal{F}_{\pm}(g)(x)||_c e^{||x||_c||y||_c}}{(1+||x||_c+||y||_c)^N} dxdy < +\infty \tag{4.7}$$

$$\int_{||x||_c\leq R}\int_{\mathbb{R}^m} ||g(x)||_c||\mathcal{F}_{\pm}(g)(y)||_c e^{||x||_c||y||_c} dxdy \leq C(1+R)^N. \tag{4.8}$$

Since $\mathcal{F}(f) \in B(\mathbb{R}^m)\otimes \mathcal{C}l_{0,m}$, (4.5) is a simple deduction from (4.4).
We shall prove (4.6).
Using (4.4) and Theorem 3.2, it follows that

$$||\mathcal{F}_{\pm}(g)(y)||_c \leq C(1+A)^{\frac{m-2}{2}}||f||_B e^{-\frac{||y||_c^2}{2}}, \quad \forall ||y||_c \leq A$$

and

$$||\mathcal{F}_{\pm}(g)(y)||_c \leq C||f||_B e^{-\frac{||y||_c^2}{4}}, \quad \forall ||y||_c > A.$$

Thus, we get (4.6).
In order to establish (4.7), we use (4.4), Definition 3.2 and Theorem 3.5.
Therefore, we find

$$\begin{aligned} I &:= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{||g(x)||_c||\mathcal{F}_{\pm}(g)(y)||_c e^{||x||_c||y||_c}}{(1+||x||_c+||y||_c)^N} dxdy \\ &\leq \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{||f(t)||_c e^{-\frac{||x-t||_c^2}{2}}||\mathcal{F}_{\pm}(f)(y)||_c e^{-\frac{||y||_c^2}{2}} e^{||x||_c||y||_c}}{(1+||x||_c+||y||_c)^N} dtdxdy \\ &\leq \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} ||f(t)||_c||\mathcal{F}_{\pm}(f)(y)||_c A(t,y) e^{||t||_c||y||_c} dtdy, \end{aligned}$$

with $A(t,y) := e^{-\frac{(||t||_c+||y||_c)^2}{2}} \displaystyle\int_{\mathbb{R}^m} \frac{e^{-\frac{||x||_c^2}{2}} e^{<x,t>} e^{||x||_c||y||_c}}{(1+||x||_c+||y||_c)^N} dx.$
We should prove that

$$A(t,y) \leq C(1+||t||_c+||y||_c)^{-N}. \tag{4.9}$$

According to the Cauchy-Schwarz's inequality, we have

$$|<x,t>| \leq ||x||_c||t||_c.$$

Thus

$$\int_{\mathbb{R}^m} \frac{e^{-\frac{||x||_c^2}{2}} e^{<x,t>} e^{||x||_c||y||_c}}{(1+||x||_c+||y||_c)^N} dx \leq e^{\frac{(||t||_c+||y||_c)^2}{2}} \int_{\mathbb{R}^m} \frac{e^{-\frac{(||x||_c-||t||_c-||y||_c)^2}{2}}}{(1+||x||_c+||y||_c)^N} dx.$$

Hence

$$A(t,y) \leq \int_{\mathbb{R}^m} \frac{e^{-\frac{(||x||_c-||t||_c-||y||_c)^2}{2}}}{(1+||x||_c+||y||_c)^N} dx.$$

Fix $0 < l < 1$. Let $B = (1+||t||_c+||y||_c)$.

$$A(t,y) \leq \int_{|||x||_c-||t||_c-||y||_c|>lB} e^{-\frac{(||x||_c-||t||_c-||y||_c)^2}{2}} dx$$

$$+\int_{|||x||_c-||t||_c-||y||_c|\leq lB}\frac{e^{-\frac{(||x||_c-||t||_c-||y||_c)^2}{2}}}{(1+||x||_c+||y||_c)^N}dx.$$

If $|||x||_c-||t||_c-||y||_c|\leq lB$, then

$$\begin{aligned}1+||x||_c+||y||_c &\geq 1+\frac{1}{2}\Big|||x||_c-||t||_c+||t||_c\Big|+||y||_c\\ &\geq 1+\frac{1}{2}(||t||_c-\Big|||x||_c-||t||_c\Big|)+||y||_c.\end{aligned}$$

Observe that

$$\Big|||x||_c-||t||_c\Big|\leq\Big|||x||_c-||t||_c-||y||_c\Big|+||y||_c.$$

Thus,

$$\begin{aligned}1+||x||_c+||y||_c &\geq \frac{1}{2}+\frac{||t||_c}{2}+\frac{||y||_c}{2}-\frac{1}{2}\Big|||x||_c-||t||_c-||y||_c\Big|\\ &\geq \frac{1}{2}B-\frac{l}{2}B\\ &\geq \frac{(1-l)}{2}B.\end{aligned}$$

We conclude (4.9).
Now, we return to the proof of (4.7) by applying (4.9) as follows

$$\begin{aligned}I &\leq \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}||f(t)||_c||\mathcal{F}_\pm(f)(y)||_c A(t,y)e^{||t||_c||y||_c}dtdy\\ &\leq C\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}||f(t)||_c||\mathcal{F}_\pm(f)(y)||_c\frac{e^{||t||_c||y||_c}}{(1+||t||_c+||y||_c)^N}dtdy.\end{aligned}$$

Subsquently, (4.7) is carried out by (4.3).

Fix $k>4$. Let

$$J:=\int_{||x||_c\leq R}\int_{\mathbb{R}^m}||g(x)||_c||\mathcal{F}_\pm(g)(y)||_c e^{||x||_c||y||_c}dxdy$$

$$=\int_{||x||_c\leq R}||g(x)||_c\left(\int_{||y||>kR}||\mathcal{F}_\pm(g)(y)||_c e^{||x||_c||y||_c}dy+\int_{||y||<kR}||\mathcal{F}_\pm(g)(y)||_c e^{||x||_c||y||_c}dy\right)dx.$$

Relation (4.6) implies

$$\begin{aligned}J &\leq \int_{||x||_c\leq R}||g(x)||_c\left(\int_{||y||>kR}Ce^{-(\frac{1}{4}-\frac{1}{k})||y||_c^2}dy+\int_{||y||<kR}||\mathcal{F}_\pm(g)(y)||_c e^{||x||_c||y||_c}dy\right)dx\\ &\leq C||g||_{1,c}+\int_{||x||_c\leq R}\int_{||y||<kR}||g(x)||_c||\mathcal{F}_\pm(g)(y)||_c e^{||x||_c||y||_c}dydx\\ &\leq C||g||_{1,c}(1+R)^N+\int_{||x||_c\leq R}\int_{||y||<kR}||g(x)||_c||\mathcal{F}_\pm(g)(y)||_c e^{||x||_c||y||_c}dydx.\end{aligned}$$

Multiplying and dividing by $(1+||x||_c+||y||_c)^N$ in the integral of right side, we obtain

$$\begin{aligned}
&\int_{||x||_c\leq R}\int_{||y||<kR}||g(x)||_c||\mathcal{F}_{\pm}(g)(y)||_c e^{||x||_c||y||_c}dydx\\
&\qquad\leq C\big(1+(k+1)R\big)^N\int_{||x||_c\leq R}\int_{||y||<kR}\frac{||g(x)||_c||\mathcal{F}_{\pm}(g)(y)||_c e^{||x||_c||y||_c}}{(1+||x||_c+||y||_c)^N}dxdy\\
&\qquad\leq C'(1+R)^N\int_{||x||_c\leq R}\int_{||y||<kR}\frac{||g(x)||_c||\mathcal{F}_{\pm}(g)(y)||_c e^{||x||_c||y||_c}}{(1+||x||_c+||y||_c)^N}dxdy,
\end{aligned}$$

where $C'=C(1+k)^N$.
Using (4.7), we complete the proof of (4.8).
Step 2. We introduce a function Γ on $\mathbb{C}\otimes\mathbb{R}^m$ as

$$\Gamma(z)=\int_0^{z_1}...\int_0^{z_m}g(u)g(ju)du.$$

In this step, we shall establish that Γ is a polynomial. Indeed, we prove that Γ is an entire function with polynomial growth. Once, we show that Γ is a polynomial, by differentiation of Γ, we prove that $g(z)g(jz)$ is a polynomial.
Since $f\in L^2(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$, according to the Theorem 3.7, we have that $g\in L^2(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$. Combining Theorem 3.3 and (4.6), we get g admits an holomorphic extension to $\mathbb{C}\otimes\mathbb{R}^m$. Moreover, using these results and Lemma 3.1, we obtain that for all $z\in\mathbb{C}\otimes\mathbb{R}^m$,

$$\begin{aligned}
||g(z)||_c&=||\mathcal{F}_{\pm}\circ\mathcal{F}_{\pm}(g)(z)||_c\\
&\leq(2\pi)^{-\frac{m}{2}}\int_{\mathbb{R}^m}e^{||y||_c||z||_c}||\mathcal{F}_{\pm}(g)(y)||_c dy\\
&\leq(2\pi)^{-\frac{m}{2}}C\int_{\mathbb{R}^m}e^{||y||_c||z||_c}e^{-\frac{||y||_c^2}{4}}dy\\
&\leq Ce^{||z||_c^2}.
\end{aligned}$$

Thus, g is entire of order 2.
Since g is entire of order 2, then Γ is entire and of order 2.

We study now the boundedness of Γ. Since $g\in L^2(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$, using Theorem 3.3, we find that

$$g(z)=\mathcal{F}_{\pm}\circ\mathcal{F}_{\pm}(g)(z).$$

Note that

$$||e^{j\theta}x||_c=||\Big(\cos(\theta)+j\sin(\theta)\Big)x||_c=||x||_c.$$

Therefore, by Lemma 3.1, it follows that for all $x\in\mathbb{R}^m$ and $\theta\in\mathbb{R}$

$$||g(e^{j\theta}x)||_c\leq C\int_{\mathbb{R}^m}e^{||x||_c||y||_c}||\mathcal{F}_{\pm}(g)(y)||_c dy. \tag{4.10}$$

Let $\epsilon\in\mathbb{R}^m$ and $\Gamma_\epsilon(\underline{z})=\Gamma(\epsilon\underline{z}),\quad\forall\underline{z}\in\mathbb{C}$. It follows that Γ_ϵ is entire function of order 2. Using (4.8) and (4.10), we obtain that Γ_ϵ is polynomial growth on $\mathbb{R}$ and $j\mathbb{R}$. In order to prove that Γ is polynomial growth in the first quadrant, we apply Phragmen-Lindelhof's Theorem (see Theorem 4.1). However, in this theorem we can't use an

angular sector of angle $\frac{\pi}{2}$. So, we will prove that Γ_ϵ is polynomial growth in all angular sectors $\{re^{j\beta},\, r \geq 0, 0 < \beta_0 \leq \beta \leq \frac{\pi}{2}\}$.
Let $0 < \alpha < \beta_0$. We consider $\Gamma^\alpha_\epsilon(\underline{z}) = \Gamma^\alpha(\epsilon \underline{z})$ with

$$\Gamma^\alpha(z) = \int_0^{z_1} \dots \int_0^{z_m} g(e^{-j\alpha}u)g(ju)du.$$

It follows from (4.8) that Γ^α_ϵ has polynomial growth on $e^{j\alpha}\mathbb{R}$ and on $j\mathbb{R}$.

Referring to Phragmen-Lindelholf's Theorem (see Theorem 4.1), we get the same estimate is valid inside the angular sector. Similarly, we show that Γ_ϵ is an entire function with polynomial growth of order N in the other three quadrants, so a polynomial of degree $\leq N$. Thus,

$$\Gamma_\epsilon(\underline{z}) = a_0(\epsilon) + a_1(\epsilon)\underline{z} + \cdots + a_N(\epsilon)\underline{z}^N,$$

Then,

$$a_k(\epsilon) = \frac{1}{k!}\frac{d^k(\Gamma(\underline{z}\epsilon))}{d\underline{z}^k}|_{\underline{z}=0},\ \forall k \in \{0, \cdots, N\}.$$

Subsquently, a_j is $\mathcal{C}l_{0,m}$-valued homogeneous polynomial on $\mathbb{R}^m$.
Since Γ is entire and polynomial on $\mathbb{R}^m$ by the principle of analytic continuation, Γ is a polynomial on $\mathbb{C} \otimes \mathbb{R}^m$. By differentiation, we get that

$$g(z)g(jz) = R(z), \tag{4.11}$$

with R is a $\mathcal{C}_m$-valued polynomial.
Step 3. By Lemma 4.2, the solution of the equation (4.11) is $g(z) = P(z)e^{Q(z)}$ with $Q(z)$ is a complex-valued polynomial of degree at most 2 and P is a $\mathcal{C}_m$-valued polynomial.
Using the fact that $e^{Q(z)}$ is a complex-valued function and equation (4.11), we find

$$Q(z) + Q(jz) = 0.$$

We deduce so that Q is homogeneous of degree 2. Finally, we obtain that

$$g(x) = P(x)e^{-b||x||_c^2}, \quad b > 0.$$

On one hand, it follows from Theorem 3.4 that

$$\mathcal{F}_\pm(g)(x) = H(x)e^{-\frac{1}{4b}||x||_c^2},$$

where H is a $\mathcal{C}_m$-valued polynomial with degree equal to the degree of P.
On the other hand, referring to (4.4), we have

$$\mathcal{F}_\pm(g)(x) = \mathcal{F}_\pm(f)(x)e^{-\frac{||x||_c^2}{2}}.$$

Therefore, we conclude that

$$\mathcal{F}_\pm(f)(x) = H(x)e^{-(\frac{1}{4b}-\frac{1}{2})||x||_c^2},$$

We recall that form the hypothesis of the Beurling's Theorem, we have that f belongs to $L^2(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$. This allow us to use Theorem 3.3 and to show that f can be written as $f(x) = Q(x)e^{-a||x||_c^2}, \quad a > 0.$ ■

5. Applications to other uncertainty principles

In this section, we show the relevance of Beurling's theorem since it entails Hardy, Cowling and Price and Gelfand Shilov theorems.

Corollary 5.1 (Hardy theorem). *Let m be even. Assume that $f \in L^2(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$ satisfies*

$$||f(x)||_c \leq C(1+||x||_c)^N e^{-a||x||_c^2} \tag{5.1}$$

and

$$||\mathcal{F}(f)(y)||_c \leq C(1+||y||_c)^N e^{-b||y||_c^2}, \tag{5.2}$$

for some $N \in \mathbb{N}$ and for some positive contants a and b. Then, three cases can occur
i) If $ab > \frac{1}{4}$, then $f = 0$.
ii) If $ab = \frac{1}{4}$, then $f = P(x)e^{-a||x||_c^2}$ with degree of $P \leq N$.
ii) If $ab < \frac{1}{4}$, there are many functions satisfying these estimates.

Proof. See that

$$(1+||x||_c+||y||_c)^{2N} \geq (1+||x||_c)^N (1+||y||_c)^N .$$

Subsquently, we obtain

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{||f(x)||_c||\mathcal{F}_\pm(f)(y)||_c}{(1+||x||_c+||y||_c)^{2N}} e^{||x||_c||y||_c} dxdy$$
$$\leq \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} e^{-(\sqrt{a}||x||_c-\sqrt{b}||y||_c)^2} e^{(1-2\sqrt{ab})||x||_c||y||_c} dxdy.$$

If $ab > \frac{1}{4}$, by Theorem 4.4, we have

$$f(x) = P(x)e^{-\delta||x||_c^2}.$$

Referring to (5.1), (5.2) and Theorem 3.4, $f = 0$ if $ab > \frac{1}{4}$.
If $ab = \frac{1}{4}$, by Theorem 4.4, $f = P(x)e^{-\delta||x||_c^2}$ with $\delta > 0$. From relations (5.1) and (5.2), δ should be equal to a and degree of $P < \frac{N-m}{2}$ which leads to the desired result.
For $ab < \frac{1}{4}$, let $f(x) = Ce^{-t||x||_c^2}$ with $t \in [a, \dfrac{1}{4b}]$ and C is a Clifford constant. Then, f satisfy the assumptions of the theorem. ∎

Corollary 5.2 (Cowling and Price theorem). *Let $N \in \mathbb{N}$, $\alpha, \beta > 0$, $1 \leq p, q < \infty$ and m be even. Let $f \in L^2(\mathbb{R}^m) \otimes \mathcal{C}l_{0,m}$ be such that*

$$\int_{\mathbb{R}^m} \frac{e^{\alpha p||x||_c^2}||f(x)||_c^p}{(1+||x||_c)^N} dx < \infty \tag{5.3}$$

$$\int_{\mathbb{R}^m} \frac{e^{\beta q||y||_c^2}||\mathcal{F}(f)(y)||_c^q}{(1+||y||_c)^N} dy < \infty. \tag{5.4}$$

Then
i) $f=0$, *if* $\alpha\beta>\frac{1}{4}$.
ii) $f=P(x)e^{-\alpha||x||_c^2}$ *whith* P *is a polynomial of degree* $< min\{\frac{N-m}{p},\frac{N-m}{q}\}$, *if* $\alpha\beta=\frac{1}{4}$.
iii) There are many functions satisfying the assumptions of the theorem, if $\alpha\beta<\frac{1}{4}$.

Proof. Let $M>max\{m+\frac{N-m}{p},m+\frac{N-m}{q}\}$.
Applying Hölder's inequality, we find that

$$\int_{\mathbb{R}^m}\frac{e^{\alpha||x||_c^2}||f(x)||_c}{(1+||x||_c)^M}dx<\infty$$

and

$$\int_{\mathbb{R}^m}\frac{e^{\beta||y||_c^2}||\mathcal{F}(f)(y)||_c}{(1+||y||_c)^M}dy<\infty.$$

Note that

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{||f(x)||_c||\mathcal{F}(f)(y)||_ce^{||x||_c||y||_c}}{(1+||x||_c)^M(1+||y||_c)^M}dxdy$$

$$=\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{||f(x)||_ce^{\alpha||x||_c^2}||\mathcal{F}(f)(y)||_ce^{\beta||y||_c^2}e^{-\left(\sqrt{\alpha}||x||_c-\sqrt{\beta}||y||_c\right)^2}e^{(1-2\sqrt{\alpha\beta})||x||_c||y||_c}}{(1+||x||_c)^M(1+||y||_c)^M}dxdy.$$

Thus, if $\alpha\beta\geq\frac{1}{4}$, we have

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{||f(x)||_c||\mathcal{F}(f)(y)||_ce^{||x||_c||y||_c}}{(1+||x||_c+||y||_c)^{2M}}dxdy<\infty.$$

Therefore, if $\alpha\beta\geq\frac{1}{4}$, by Theorem 4.4,

$$f(x)=Q(x)e^{-\delta||x||_c^2},\delta>0.$$

with the degree of the polynomial Q is less than $M-\dfrac{m}{2}$. Returning to the conditions of the corollary, if $\alpha\beta>\frac{1}{4}$, $f=0$. Furthermore, if $\alpha\beta=\frac{1}{4}$, $\delta=\alpha$ and the degree of $Q<min\{\frac{N-m}{p},\frac{N-m}{q}\}$.
If $\alpha\beta<\frac{1}{4}$, we can take the same example used in Corollary 5.1.

∎

Corollary 5.3 (Gelfand Shilov theorem)**.** *Let* $N\in\mathbb{N}$, $1<p,q<\infty$, $\alpha,\beta>0$ *and* m *be even. Let* $f\in L^2(\mathbb{R}^m)\otimes\mathcal{C}l_{0,m}$ *satisfy*

$$\int_{\mathbb{R}^m}\frac{||f(x)||_ce^{\frac{(2\alpha||x||_c)^p}{p}}}{(1+||x||_c)^N}dx<\infty \tag{5.5}$$

$$\int_{\mathbb{R}^m}\frac{||\mathcal{F}(f)(y)||_ce^{\frac{(2\beta||y||_c)^q}{q}}}{(1+||y||_c)^N}dy<\infty \tag{5.6}$$

with $\alpha\beta\geq\frac{1}{4}$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *We have the following results :*
i) If $\alpha\beta>\frac{1}{4}$ *or* $(p,q)\neq(2,2)$, *then* $f=0$ *almost everywhere.*

ii) If $\alpha\beta = \frac{1}{4}$ *and* $p = q = 2$*, then* $f = P(x)e^{-2\alpha^2||x||_c^2}$ *where* P *is a polynomial with degree less than* $N - m$*.*

Proof. Using (5.5) and (5.6) and the fact that

$$4\alpha\beta||x||_c||y||_c \leq \frac{(2\alpha)^p}{p}||x||_c^p + \frac{(2\beta)^q}{q}||y||_c^q,$$

we get

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{||f(x)||_c||\mathcal{F}_\pm(f)(y)||_c}{(1+||x||_c+||y||_c)^{2N}} e^{4\alpha\beta||x||_c||y||_c} dxdy < \infty. \tag{5.7}$$

Since $\alpha\beta \geq \frac{1}{4}$, it follows from Theorem 4.4 that

$$f(x) = P(x)e^{-a||x||_c^2}$$

with degree of P less than $\dfrac{N-m}{2}$. Thus, by Theorem 3.4, (5.7) can be written as

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{||P(x)||_c e^{-\left(\sqrt{a}||x||_c - \frac{||y||_c}{2\sqrt{a}}\right)^2}||Q(y)||_c}{(1+||x||_c+||y||_c)^{2N}} e^{(4\alpha\beta-1)||x||_c||y||_c} dxdy < \infty$$

When $\alpha\beta > \frac{1}{4}$, this integral is not finite only if $f = 0$ almost everywhere.
Moreover, see that (5.5) and (5.6) are satisfied only when $(p, q) = (2, 2)$.
When $\alpha\beta = \frac{1}{4}$ and $(p, q) = (2, 2)$, (5.5) and (5.6) imply that the degree of $P < N - m$ and $a = 2\alpha^2$. ∎

6. Conclusion

In the present paper, we established uncertainty principles in the setting of the Clifford analysis. These principles state that a Clifford-valued function and its Clifford-Fourier transform cannot be simultaneously sharply localized. We started by proving the Beurling's theorem for the Clifford-Fourier transform which is the most interesting theorem since it implies other well known uncertainty principles. Then, we derived some applications such as the Hardy uncertainty principle, Cowling and Price uncertainty principle, and Gelfand-Shilov uncertainty principle. We should mention that the Hardy uncertainty principle was proved in another paper differently. However, Cowling and Price uncertainty principle and Gelfand-Shilov uncertainty principle are new results. Furthermore, we recall that the Clifford-Fourier transform presents a generalization of other Fourier transforms such as the quaternion-Fourier transform and the Hankel-Fourier transform. Regarding this, we can obtain Beurling, Hardy, Cowling and Price and Gelfand-Shilov uncertainty principles for all these Fourier transforms.

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