Arc Entropy of Uncertain Variables and Its Applications

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Abstract

The problem of quantifying uncertainty has not been well solved. To measure the uncertainty of uncertain variables, we first propose the concept of arc entropy via uncertainty distributions and introduce a new effective method in this paper. Some properties of arc entropy are derived, and some practical examples of uncertain variables are given. A formula for arc entropy is derived via inverse uncertainty distributions, and several basic theorems are proposed. Moreover, two general arc entropies are defined, and their properties are investigated. An application to uncertain learning curves is introduced, and an uncertain learning curve model is proposed. Another application to portfolio selection is presented, and its mathematical model is established.

Keywords: uncertain variable, arc entropy, portfolio selection, learning curve

1 Introduction

Entropy, as a standard tool to measure the uncertainty for a random variable in real life, was first investigated by Shannon [13] in 1949. Hence, Shannon entropy was the inauguration of information theory in the uncertainty field. Today, it is common to find Shannon entropy in several models. However, in many cases, this entropy is almost nonexistent. In recent years, extensive research on entropy has been conducted in many areas. Various entropies have been widely introduced and applied in probability theory, fuzzy theory and other theories. Entropy is constantly being innovated and developed, and it plays an active role in measuring the uncertainty of a variable in uncertain phenomena and systems. For example, entropy in fuzzy theory, as a supplement measuring the information deficiency associated with a fuzzy event, was presented by Zadeh [21] in 1968 for a random variable. Moreover, quadratic entropy was presented by Vaida [11] to measure the degree of uncertain information for a random variable in 1968.

To deal with degrees of belief for uncertain problems of real-world information, an uncertainty theory was proposed by Liu [6] in 2007 and perfected by Liu [7] with an uncertainty measure in 2009. Currently, uncertainty theory has become a branch of mathematics concerned with the analysis of belief degrees,

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and it has been studied by many researchers. The core of uncertainty theory is an uncertainty measure to deal with human uncertainty. Based on uncertainty measures, a definition of uncertain variables was given by Liu [6] to describe uncertain phenomena or uncertain problems in 2007. The entropy was proposed by Liu [7] in a logarithmic form to handle uncertainty information in 2009. Subsequently, many researchers have proposed various entropies for uncertain variables. For example, Chen and Dai [1] presented entropy for uncertain variable and gave some mathematical properties such as the maximum entropy principle in 2011. Dai and Chen [15] proposed some formulas to compute entropy via inverse uncertainty distributions and further proved some properties such as the positive linearity property in 2012. Chen et al.[3] introduced cross entropy used to measure the divergence of variables and proposed the minimum cross entropy principle in 2012. Yao et al. [17] proposed the concept of the sine entropy of uncertain variables and proved the maximum sine entropy with given expectation and variance in 2013. Tang and Gao [14] studied the triangular entropy of uncertain variables in 2013. Dai proposed the concept of quadratic entropy and proved the value range of entropy in 2017. Gao et al. [5] gave a new definition of cross entropy and further derived a formula for cross entropy via inverse uncertainty distributions in 2018. Moreover, entropy has been applied to many uncertain phenomena and uncertain problems. For instance, Zhang and Meng [22] proposed an expected-variance entropy model and gave a numerical example in 2012. Ning et al. [12] proposed a triangular entropy model for an uncertain portfolio market. Entropy has been proven to be an effective tool for dealing with uncertain data and uncertain phenomena.

In this study, we propose arc entropy for uncertain variables as an effective tool for investigating its main properties and its applications in many areas. The rest of this paper is organized as follows: The next section is intended to briefly introduce some useful concepts of uncertain variables and the research situation of various entropies. Section 3 first proposes the concept of arc entropy via an uncertainty distribution and investigates its properties. A key formula is presented to calculate arc entropy via an inverse distribution function, and several basic theorems are proposed in Section 4. Two general arc entropies are introduced, and their mathematical properties are presented in Section 5. An application to an uncertain learning curve is introduced, and a basic formula is obtained in Section 6. An application to portfolio selection is proposed, and a portfolio selection model is established in Section 7. Finally, this paper is concluded in Section 8.

2 Preliminaries

This section gives some basic definitions, preliminary concepts and essential theorems about the uncertainty theory founded by Liu [6].

Let Γ be a nonempty set and $\mathcal{L}$ be a σ-algebra over Γ. A set function $\mathcal{M}$: $\mathcal{L} \to [0,1]$ is called an uncertainty measure if and only if it satisfies the following four axioms:

$Axiom 1$: (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;

$Axiom 2:$ (Self-Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;

Axiom 3: (Subadditivity Axiom) For every countable event sequence $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\};$$

$Axiom 4:$ (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be the uncertainty space for $k = 1, 2, \cdots, n$. The product uncertainty measure $\mathcal{M}$ is defined as follows:

$$\mathcal{M}\left\{\prod_{k=1}^{n} \Lambda_k\right\} = \min_{1\leq k\leq n} \mathcal{M}_k\{\Lambda_k\}.$$

Definition 1. *(Liu 2010 [8]) Let ξ be an uncertain variable with a regular uncertainty distribution $\Phi(x)$, denoted by $\xi \sim (\xi, \Phi(x))$. Then, the function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ, denoted by $\xi \sim (\xi, \Phi^{-1}(x))$.*

Example 1. *A linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ with an uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ \dfrac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } x \geq b \end{cases}$$

has its inverse uncertainty distribution as follows:

$$\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b.$$

Example 2. *A zigzag variable $\xi \sim \mathcal{Z}(a, b, c)$ with an uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ \dfrac{x-a}{2(b-a)}, & \text{if } a \leq x \leq b \\ \dfrac{x+c-2b}{2(c-b)}, & \text{if } b \leq x \leq c \\ 1, & \text{if } x \geq c \end{cases}$$

has its inverse distribution as follows:

$$\Phi^{-1}(\alpha) = \begin{cases} a + 2(b-a)\alpha, & \text{if } \alpha \leq 0.5 \\ 2b - c + 2(c-b)\alpha, & \text{if } \alpha \geq 0.5. \end{cases}$$

Theorem 1. *(Liu 2010 [9]) Based on the above definition, the expected value of the variable ξ is*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha. \tag{1}$$

The variance of the variable ξ is given as follows:

$$V[\xi] = \int_0^1 (\Phi^{-1}(\alpha) - E[\xi])^2 d\alpha. \tag{2}$$

We recall some concepts of entropy proposed by scholars for handling uncertain problems in recent years.

Definition 2. *(Liu 2009 [7] Let $\xi \sim (\xi, \Phi(x))$. Then, the entropy of ξ is given by*

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))dx,$$

Definition 3. *(Dai [16]) Let $\xi \sim (\xi, \Phi(x))$. Then, the quadratic entropy of ξ is defined by*

$$Q[\xi] = \int_{-\infty}^{+\infty} \Phi(t)(1 - \Phi(t))dt.$$

Definition 4. *(Chen 2011 [2]) Let $\xi \sim (\xi, \Phi(x))$. The cross entropy of ξ from η is introduced by*

$$D[\xi : \eta] = \int_{-\infty}^{+\infty} T(\mathcal{M}\{\xi \leq x\}, \mathcal{M}\{\eta \leq x\})dx.$$

Definition 5. *(Dai and Chen 2012 [15]) Let $\xi \sim (\xi, \Phi^{-1}(x))$. The entropy of ξ is express by*

$$H[\xi] = \int_{0}^{1} \Phi^{-1}(\alpha) ln \frac{\alpha}{1-\alpha} dx.$$

Definition 6. *(Tang and Gao 2013 [14]) Let $\xi \sim (\xi, \Phi(x))$. The triangular entropy of ξ is proposed by*

$$T[\xi] = \int_{-\infty}^{+\infty} K(\Phi(x))dx.$$

Definition 7. *(Yao 2013 [17]) Let $\xi \sim (\xi, \Phi(x))$. The sine entropy of ξ is expressed by*

$$S[\xi] = \int_{-\infty}^{+\infty} sin(\pi\mu(x))dx.$$

Theorem 2. *(Liu 2007 [6]) Let independent uncertain variables $\xi_i \sim \Phi_i(x)$. If $f(x_1, x_2, \cdots, x_n)$ is a strictly increasing function with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable with inverse uncertainty distribution*

$$\Upsilon^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Definition 8. *(Li 2019 [10]) Let the production cost be an uncertain variable denoted by*

$$C = KA^{\eta},$$

where the variable C is the cost of the A-th unit with $A \in N$. The parameter K is assumed cost of the first unit with a constant $K \in R^{+}$. The uncertain parameter $\eta \in [-1, 0]$ is a learning variable following an uncertainty distribution $\Phi(x)$.

Theorem 3. *(Li 2019 [10]) Let the cost be defined as in Definition 8. Then, the expected value of the cost is*

$$E[C] = \int_{0}^{1} KA^{\Phi^{-1}(\alpha)} d\alpha.$$

3 Arc Entropy and Its Properties

This section first proposes the concept of arc entropy for uncertain variables with uncertainty distributions. Some properties of arc entropy are studied, and some practical examples of the arc entropy are given.

Definition 9. *Let $\xi \sim (\xi, \Phi(x))$. Then, its arc entropy is defined by*

$$A[\xi] = \int_{-\infty}^{\infty} arc(\Phi(x))dx = \int_{-\infty}^{\infty} \sqrt{1 - \Phi(x)^2 + \Phi(x)} - 1dx.$$

Clearly, the function $\mu(t) = \sqrt{1-t^2+t} - 1 \quad (0 \le t \le 1)$ is the upper part of an arc curve and is a symmetric function. It is firstly increasing and then decreasing. Its unique maximum $\mu(0.5) = \frac{\sqrt{5}}{2} - 1$.

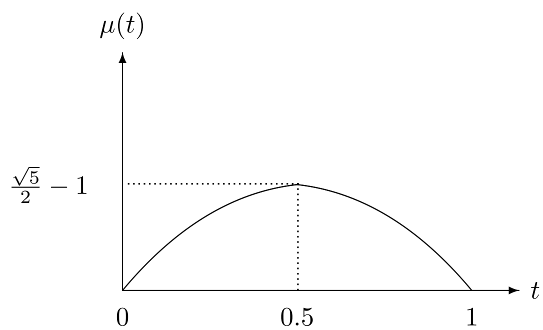

Fig. 1. $\mu(t) = \sqrt{1-t^2+t} - 1$

Example 3. *Let a variable $\xi \sim (\xi, c(x))$ as*

$$c(x) = \begin{cases} 0, & \text{if } x \le c \\ 1, & \text{if } x \ge c. \end{cases}$$

From Definition 9, the arc entropy of ξ can be calculated as follows:

$$A[\xi] = \int_{-\infty}^{c} \sqrt{1-0^2+0} - 1dx + \int_{c}^{+\infty} \sqrt{1-1^2+1} - 1dx = 0.$$

That is, a constant has no uncertainty.

Example 4. *Let $\xi \sim \mathcal{L}(a,b)$. $A[\xi]$ can be given as*

$$A[\xi] = \int_a^b \sqrt{1 - (\frac{x-a}{b-a})^2 + \frac{x-a}{b-a}} - 1dx = \frac{5\arcsin\frac{\sqrt{5}}{5} - 2}{4}(b-a).$$

Example 5. *Let $\xi \sim \mathcal{Z}(a,b,c)$. $A[\xi]$ can be calculated as*

$$\begin{aligned} A[\xi] &= \int_{-\infty}^{a} \sqrt{1-0^2+0} - 1dx + \int_a^b \sqrt{1 - (\frac{x-a}{2(b-a)})^2 + \frac{x-a}{2(b-a)}} - 1dx \\ &+ \int_b^c \sqrt{1 - (\frac{x+c-2b}{2(c-b)})^2 + \frac{x+c-2b}{2(c-b)}} - 1dx + \int_c^{+\infty} \sqrt{1-1^2+1} - 1dx \\ &= \frac{5\arcsin\frac{\sqrt{5}}{5} - 2}{4}(c-a). \end{aligned}$$

Arc entropy is proposed via the arc function and provides a useful tool as a supplement to measure the uncertainty in real world.

Theorem 4. *Arc entropy of* $\xi \sim (\xi, \Phi(x))$ *is* $A[\xi] \geq 0$. $A[\xi] = 0$ *if* $\Phi(x) = 0$ *or* $\Phi(x) = 1$.

Proof: Clearly, arc entropy is nonnegative. In fact, when the distribution $\Phi(x)$ tends towards 0 or 1, $A[\xi]$ approaches the minimum value 0.

Theorem 5. *(Principle of maximum arc entropy) Arc entropy of* $\xi \sim (\xi, \Phi(x))$ *on [a, b] is*

$$A[\xi] \leq (\frac{\sqrt{5}}{2} - 1)(b - a),$$

and $A[\xi] = (\frac{\sqrt{5}}{2} - 1)(b - a)$ *if* $\Phi(x) = 0.5$.

Proof: We know that the function $\mu(t) = \sqrt{1 - t^2 + t} - 1$ is an arc function opening downward. In fact, its maximum value $\mu(0.5) = \dfrac{\sqrt{5}}{2} - 1$. Then,

$$\begin{aligned} A[\xi] &= \int_a^b arc(\Phi(x))dx = \int_a^b \sqrt{1 - (\Phi(x))^2 - \Phi(x)} - 1dx \\ &\leq \int_a^b \frac{\sqrt{5}}{2} - 1dx = (\frac{\sqrt{5}}{2} - 1)(b - a). \end{aligned}$$

Theorem 6. *(Translation Invariance) Let* $\xi \sim (\xi, \Phi(x))$*, and a real number* $k \in R$*. Then, we have*

$$A[\xi + k] = A[\xi].$$

Proof: Denote $\xi \sim (\xi, \Phi(x))$. Then, $\xi + k \sim (\xi + k, \Phi(x + k))$. From the Definition 9, we have

$$\begin{aligned} A[\xi + k] &= \int_{-\infty}^{+\infty} arc(\Phi(x + k))dx \\ &= \int_{-\infty}^{+\infty} \sqrt{1 - (\Phi(x + k))^2 + \Phi(x + k)} - 1d(x + k) = \int_{-\infty}^{+\infty} arc(\Phi(x))dx = A[\xi]. \end{aligned}$$

The theorem is verified.

4 A Key Formula and Its Theorems

Here, a formula for arc entropy is derived. Several basic theorems are investigated, and some practical examples of arc entropy are given.

Theorem 7. *Let* $\xi \sim (\xi, \Phi^{-1}(x))$*. If the arc entropy* $A(\Phi(x))$ *exists, then*

$$A[\xi] = \int_0^1 \Phi^{-1}(\alpha) \frac{2\alpha - 1}{2\sqrt{1 - \alpha^2 + \alpha}} d\alpha.$$

Proof: Let $f(\alpha) = \sqrt{1 - \alpha^2 + \alpha} - 1$. Then, a derivable function of $f(\alpha)$ is $f'(\alpha) = \dfrac{1 - 2\alpha}{2\sqrt{1 - \alpha^2 + \alpha}}$.

Since $f(\Phi(x)) = \displaystyle\int_0^{\Phi(x)} f'(\alpha)d\alpha = -\int_{\Phi(x)}^1 f'(\alpha)d\alpha$, it follows from Definition 9 that

$$A[\xi] = \int_{\infty}^{+\infty} f(\Phi(x))dx = \int_{-\infty}^0 \int_0^{\Phi(x)} f'(\alpha)d\alpha dx - \int_0^{+\infty} \int_{\Phi(x)}^1 f'(\alpha)d\alpha dx.$$

From the Fubini theorem, we have

$$
\begin{aligned}
A[\xi] &= \int_0^{\Phi(0)} \int_{\Phi^{-1}(\alpha)}^{0} f'(\alpha) dx d\alpha - \int_{\Phi(0)}^{1} \int_0^{\Phi^{-1}(\alpha)} f'(\alpha) d\alpha \\
&= - \int_0^{\Phi(0)} \Phi^{-1}(\alpha) f'(\alpha) dx d\alpha - \int_{\Phi(0)}^{1} \Phi^{-1}(\alpha) f'(\alpha) d\alpha \\
&= - \int_0^1 \Phi^{-1}(\alpha) f'(\alpha) d\alpha \\
&= \int_0^1 \Phi^{-1}(\alpha) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2+\alpha}} d\alpha.
\end{aligned}
$$

The theorem is thus verified.

Example 6. *Let $\xi \sim (\xi, \Phi^{-1}(x))$ with $\Phi^{-1}(\alpha) = \alpha$. The arc entropy of ξ can be calculated as follows:*

$$
A[\xi] \;= \int_0^1 \Phi^{-1}(\alpha) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2+\alpha}} d\alpha = \frac{5 \arcsin \frac{\sqrt{5}}{5} - 2}{4}.
$$

Theorem 8. *Let $\xi \sim (\xi, \Phi^{-1}(x))$ and $\varphi \sim (\varphi, \Psi^{-1}(x))$, respectively. Then,*

$$
A[a\xi + b\varphi] = |a| A[\xi] + |b| A[\varphi]
$$

for $a, b \in R$.

Proof: This theorem will be proved via certain assumptions as follows:

First, we prove $A[a\xi] = |a| A[\xi]$.

If $a > 0$, then $a\xi \sim (a\xi, \Upsilon^{-1}(\alpha)) = (a\xi, a\Phi^{-1}(\alpha))$.

Following Theorem 7, we have

$$
A[a\xi] = \int_0^1 \Upsilon^{-1}(\alpha) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2+\alpha}} d\alpha = \int_0^1 a\Phi^{-1}(\alpha) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2+\alpha}} d\alpha = |a| A[\xi].
$$

If $a < 0$, then $a\xi \sim (a\xi, \Upsilon^{-1}(\alpha)) = (a\xi, a\Phi^{-1}(1-\alpha))$. Following Theorem 7, we have

$$
A[a\xi] = \int_0^1 a\Phi^{-1}(1-\alpha) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2+\alpha}} d\alpha = \int_0^1 (-a)\Phi^{-1}(\alpha) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2+\alpha}} d\alpha = |a| A[\xi].
$$

If $a = 0$, then $A[a\xi] = 0 = |a| A[\xi]$. Then, we have $A[a\xi] = |a| A[\xi]$.

Second, we prove that $A[\xi + \varphi] = A[\xi] + A[\varphi]$.

Following Theorem 2, $\xi + \varphi \sim (\xi + \varphi, \Upsilon^{-1}(\alpha)) = (\xi + \varphi, \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha))$.

It follows from Theorem 7 that

$$
A[\xi + \varphi] = \int_0^1 (\Phi^{-1}(\alpha) + \Psi^{-1}(\alpha)) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2+\alpha}} d\alpha = A[\xi] + A[\varphi].
$$

Finally, for $a, b \in R$, by the above proof, we have

$$
A[a\xi + b\varphi] = |a| A[\xi] + |b| A[\varphi].
$$

The proof is thus completed.

Theorem 9. *Follow Theorem 2, The arc entropy of $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is*

$$A[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2} - \alpha} d\alpha.$$

Proof: Follow Theorem 2 and Theorem 7, we have

$$\begin{aligned} A[\xi] &= \int_0^1 \Upsilon^{-1}(\alpha) \frac{2\alpha - 1}{\sqrt{1-\alpha^2} + \alpha} d\alpha \\ &= \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2} + \alpha} d\alpha. \end{aligned}$$

The theorem is verified.

Corollary 1. *If $f(x_1, x_2, \cdots, x_n)$ is a strictly increasing function, then*

$$A[\xi] = A[f(\xi_1, \xi_2, \cdots, \xi_n] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2} + \alpha} d\alpha.$$

Example 7. *Let $\xi \sim (\xi, \Phi^{-1}(\alpha)) = \xi \sim (\xi, (1-\alpha)a + \alpha b)$ be a strictly increasing function. It follows from Theorem 7 and Corollary 1 that the arc entropy of ξ is*

$$\begin{aligned} A[\xi] &= \int_0^1 \Phi^{-1}(\alpha) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2} + \alpha} d\alpha \\ &= \int_0^1 \big((1-\alpha)a + \alpha b\big) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2} + \alpha} d\alpha \\ &= \frac{5 \arcsin \frac{\sqrt{5}}{5} - 2}{4} (b - a). \end{aligned}$$

Corollary 2. *If $f(x_1, x_2, \cdots, x_n)$ is a strictly decreasing function, then*

$$A[\xi] = A[f(\xi_1, \xi_2, \cdots, \xi_n)] = \int_0^1 f(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2} + \alpha} d\alpha.$$

Example 8. *Let $\xi \sim (\xi, \Phi^{-1}(x))$ and $\varphi \sim (\varphi, \Psi^{-1}(x))$, respectively. The variables ξ and η are strictly decreasing, respectively. Then,*

$$A[\xi + \eta] = \int_0^1 (\Phi^{-1}(1-\alpha) + \Upsilon^{-1}(1-\alpha)) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2} + \alpha} d\alpha.$$

The arc entropy of $\xi\eta$ is

$$A[\xi\eta] = \int_0^1 \Phi^{-1}(1-\alpha)\Upsilon^{-1}(1-\alpha) \frac{2\alpha - 1}{2\sqrt{1-\alpha^2} + \alpha} d\alpha.$$

The arc entropy of ξ/η is

$$A\Big[\frac{\xi}{\eta}\Big] = \int_0^1 \frac{\Phi^{-1}(1-\alpha)}{\Upsilon^{-1}(1-\alpha)} \frac{2\alpha - 1}{2\sqrt{1-\alpha^2} + \alpha} d\alpha.$$

5 Two General Arc Entropies

This section introduces two generalized definitions of arc entropy, and then, their properties are investigated. Furthermore, some practical examples of the two general arc entropies are given.

Definition 10. *Let $\xi \sim (\xi, \Phi(x))$. Then, one type of general arc entropy is defined by*

$$A_1[\xi] = \int_{-\infty}^{\infty} arc(\Phi(x))dx = \int_{-\infty}^{\infty} \sqrt{a^2 - \Phi(x)^2 + \Phi(x)} - adx,$$

where $a > 0$ is an uncertain parameter.

Definition 11. *Let $\xi \sim (\xi, \Phi(x))$. Then, the other type of general arc entropy is defined by*

$$A_2[\xi] = k\int_{-\infty}^{\infty} arc(\Phi(x))dx = k\int_{-\infty}^{\infty} \sqrt{a^2 - \Phi(x)^2 + \Phi(x)} - adx,$$

where $k > 0$ and $a > 0$ are uncertain parameters.

It is clear that $h(t, a, k) = k(\sqrt{a - t^2 + t} - a)$ $(0 \le t \le 1, a > 0, k > 0)$ is the upper part of an arc curve and opens down. Note that its unique maximum $h(0.5, a, k) = k(\sqrt{a + 0.25} - a)$.

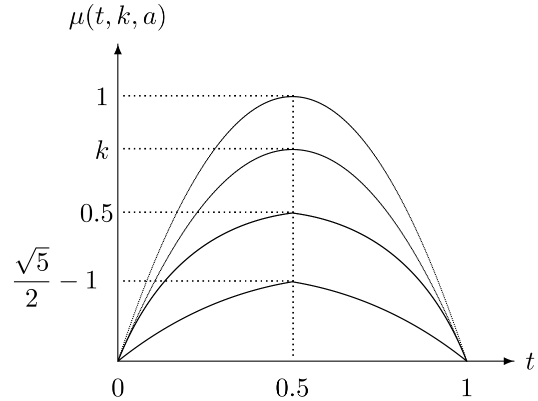

Fig. 2. $\mu(t, k, a) = k(\sqrt{a^2 - t^2 + t} - a)$

Corollary 3. *Let $\xi \sim (\xi, \Phi^{-1}(x))$. Based on Theorem 7 and Definitions 10 and Definitions 11, the two general arc entropies are given as*

$$A_1[\xi] = \int_0^1 \Phi^{-1}(\alpha)\frac{2\alpha - 1}{2\sqrt{a^2 - \alpha^2 + \alpha}}d\alpha$$

$$A_2[\xi] = k\int_0^1 \Phi^{-1}(\alpha)\frac{2\alpha - 1}{2\sqrt{a^2 - \alpha^2 + \alpha}}d\alpha,$$

respectively.

To compute entropy more conveniently, we need to simplify the formula of the general arc entropy. Namely, if uncertain parameters k and a are given, then the general arc entropy becomes a formula for arc entropy. Then, various entropies can be calculated by using Definition 10, Definition 11, and Corollary 3. For example, let $k = 4$ and $a = 2$; the general arc entropy is

$$A[\xi] = 4\int_{-\infty}^{\infty} arc(\Phi(x))dx = 4\int_{-\infty}^{\infty} \sqrt{2^2 - \Phi(x)^2 + \Phi(x)} - 2dx.$$

Example 9. *Let $\xi \sim (\xi, \Phi^{-1}(x))$ with $\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b$. Following the above formula, the arc entropy is*

$$\begin{aligned} A[\xi] &= 4\int_0^1 \Phi^{-1}(\alpha)\frac{2\alpha - 1}{2\sqrt{2^2 - \alpha^2 + \alpha}}d\alpha \\ &= (17\arcsin\frac{\sqrt{17}}{17} - 4)(b - a). \end{aligned}$$

Example 10. *Let $\xi \sim (\xi, \Phi(x))$ with $\xi \sim \mathcal{Z}(a,b,c)$. Then, the arc entropy of ξ is*

$$\begin{aligned} A[\xi] &= 4\int_a^b \sqrt{2^2 - (\frac{x-a}{2(b-a)})^2 + \frac{x-a}{2(b-a)}} - 2dx + 4\int_b^c \sqrt{2^2 - (\frac{x+c-2b}{2(c-b)})^2 + \frac{x+c-2b}{2(c-b)}} - 2dx \\ &= (17\arcsin\frac{\sqrt{17}}{17} - 4)(c-a). \end{aligned}$$

We know that the arc function is a very important function, such as a circle function, elliptic function or hyperbolic function. It is closely related to the arc curve, arc function, arc differential and differential integral, which are widely applied in the information field. To address more complicated uncertainty questions, two general arc entropies are proposed via the definition of arc entropy. A new method is developed to handle uncertainty in real life.

6 Applications to Uncertain Learning Curves

The learning curve is a mathematical tool used to estimate either the production cost or time by establishing a mathematical model. The learning parameter plays an important role in the learning curve. However, it is a constant. To measure the uncertainty of the learning curve environment, we treat the learning parameter as an uncertain variable. This section presents a new mathematical form of the arc entropy of uncertain learning curve variables.

Theorem 10. *Let ξ be an uncertain production cost event and η be an uncertain learning curve variable with $\eta \sim \Phi^{-1}(\alpha)$. Then, the arc entropy of the uncertain learning curve variables is defined by*

$$A[\xi] = K\int_0^1 f(\Phi^{-1}(\alpha))\frac{2\alpha - 1}{2\sqrt{1-\alpha^2+\alpha}}d\alpha = K\int_0^1 C^{\Phi^{-1}(\alpha)}\frac{2\alpha-1}{2\sqrt{1-\alpha^2+\alpha}}d\alpha,$$

where the uncertain production cost event $\xi = KC^\eta$ is the uncertain cost of the C-th unit with respect to η, with $C > 1$. The parameter K is assumed cost of the first unit with a constant $K \in R^+$. The uncertain parameter $\eta \in [-1, 0]$ is a learning variable with $\eta \sim \Phi^{-1}(\alpha)$.

Proof: Denote that $\xi = KC^\eta$ is a strictly increasing function of the uncertain learning curve variable η. The uncertain variable $\eta \sim \Phi^{-1}(\alpha)$. Following Theorem 1, Corollary 1 and Corollary 3, then,

$$A[\xi] = K\int_0^1 f(\Phi^{-1}(\alpha))\frac{2\alpha - 1}{2\sqrt{1-\alpha^2+\alpha}}d\alpha = K\int_0^1 C^{\Phi^{-1}(\alpha)}\frac{2\alpha-1}{2\sqrt{1-\alpha^2+\alpha}}d\alpha.$$

The theorem is verified.

Example 11. *Let an uncertain learning curve variable $\eta \sim \mathcal{L}(a,b) = \mathcal{L}(-0.6, -0.1)$. Then,*

$$\Phi^{-1}(\alpha) = -0.6(1-\alpha) - 0.1\alpha = 0.5\alpha - 0.6.$$

Therefore, following Theorem 10, then the arc entropy is

$$A[\xi] = K\int_0^1 C^{0.5\alpha - 0.6}\frac{2\alpha-1}{2\sqrt{1-\alpha^2+\alpha}}d\alpha.$$

Without loss of generality, for simplicity, we let $K = 20$ and $C = 2$; then, $A[\xi] \approx 0.43$.

Example 12. *Let an uncertain learning curve variable* $\eta \sim \mathcal{Z}(a,b,c) = \mathcal{Z}(-0.9,-0.4,-0.1)$. *Then,*

$$\Phi^{-1}(\alpha) = \begin{cases} \alpha - 0.9, & \text{if } \alpha < 0.5 \\ 0.6\alpha - 0.7, & \text{if } \alpha \geq 0.5. \end{cases}$$

Therefore, it follows from Theorem 10 that the arc entropy is

$$A[\xi] = K\int_0^{0.5} C^{\alpha-0.9}\frac{2\alpha-1}{2\sqrt{1-\alpha^2+\alpha}}d\alpha + K\int_{0.5}^{1} C^{0.6\alpha-0.7}\frac{2\alpha-1}{2\sqrt{1-\alpha^2+\alpha}}d\alpha.$$

For simplicity, let $K = 20$ *and* $C = 2$*; then,* $A[\xi] \approx 0.64$.

Theorem 11. *Let* ξ *be an uncertain production cost event and an uncertain learning curve variable* $\eta \sim \Phi^{-1}(\alpha)$. *Then, the general arc entropy of the uncertain learning curve variables is defined by*

$$A[\xi] = K\int_0^{1} f(\Phi^{-1}(\alpha))\frac{2\alpha-1}{2\sqrt{a^2-\alpha^2+\alpha}}d\alpha = K\int_0^{1} C^{\Phi^{-1}(\alpha)}\frac{2\alpha-1}{2\sqrt{a^2-\alpha^2+\alpha}}d\alpha,$$

where a*,* K *and* C *are parameters. Let* $\xi = KC^{\eta}$ *be an uncertain production cost event for the uncertain learning curve variable* η. C *is the uncertain cost of the* C*-th unit with respect to* η*, with* $C > 1$. *The parameter* K *is assumed cost of the first unit with a constant* $K \in R^+$*, and* a *is a constant of reflection for the uncertain learning curve, with* $a > 0$.

The mathematical formulas of arc entropy for uncertain learning curve variables are obtained. They are a useful tool to study the uncertainty of uncertain learning curve variables with inverse uncertainty distributions. Therefore, we can use mathematical formulas to analyse an uncertain learning curve or measure an uncertainty problem in the machine-scheduling field in the future.

7 Applications to Portfolio Selection

Portfolio selection has always been a popular and difficult problem in many theoretical studies. Yu [19] introduced a stock model of portfolio areas in 2012. At almost the same time, Yu [20] proposed the expected payoff for uncertain markets in 2012. The above research is based on a class of stocks. Suppose there are n stocks for investors to choose from in a financial market.

To clearly describe the portfolio selection problem with a mathematical model, let us first introduce the following notation:

i	$securities\ i = 1, 2, \cdots, n$
ξ_i	$return\ rate\ of\ security\ i$
k_i	$investment\ proportions\ of\ stock\ i$
a	$predetermined\ level\ of\ investment\ risk$
b	$predetermined\ level\ of\ arc\ entropy$
E	$expected\ value$
V	$variance$
$\xi_1, \xi_2, \cdots, \xi_n$	$portfolio\ selection$
$k_1, k_2, \cdots, k_n$	$investment\ proportions\ of\ stock\ n$

Every investor wants the maximal investment return with minimum risk. The expected value criterion is the most effective method for modelling in an uncertain environment. By maximizing the final wealth in the sense of expected value, an uncertain portfolio selection model can be introduced as follows:

$$\begin{cases} \underset{k_i}{max} \quad E[k_1\xi_1 + k_2\xi + \cdots + k_n\xi_n] \\ subject\ to: \\ V[k_1\xi_1 + k_2\xi + \cdots + k_n\xi_n] \leq a \\ A[k_1\xi_1 + k_2\xi + \cdots + k_n\xi_n] \geq b \\ \quad k_1 + k_2 + \cdots + k_n = 1 \\ \quad k_i \geq 0, i = 1, 2, \cdots, n, \end{cases}$$

To effectively solve the introduced model, we should ensure that the smaller the variance is, the higher the entropy; then, $E[k_1\xi_1 + k_2\xi + \cdots + k_n\xi]$ is the maximum. Hence, we will discuss the corresponding equivalents of the model. Following from Theorem 1, Theorem 7 and Theorem 8, these equivalents of the model may be expressed as:

$$\begin{cases} \underset{k_i}{max} \quad \sum_{i=1}^{n} k_i \int_0^1 \Phi_i^{-1}(\alpha)d\alpha \\ subject\ to: \\ \int_0^1 \Big(\sum_{i=1}^{n} x_i\Phi_i^{-1}(\alpha)\Big)^2 d\alpha - \Big(\sum_{i=1}^{n} k_i \int_0^1 \Phi_i^{-1}(\alpha)d\alpha\Big)^2 \leq a \\ \sum_{i=1}^{n} k_i \int_0^1 \Phi_i^{-1}(\alpha)\frac{2\alpha - 1}{\sqrt{1 + (2\alpha - 1)^2}}d\alpha \geq b \\ \quad k_1 + k_2 + \cdots + k_n = 1 \\ \quad k_i \geq 0, i = 1, 2, \cdots, n, \end{cases}$$

Let $n = 4$; then, there are four stocks selected by the investor in the financial market. For example, let the investment returns be $\xi_1 = (1, 3)$ with $\Phi^{-1}(\alpha) = 1 + 2\alpha$, $\xi_2 = (2, 4)$ with $\Phi^{-1}(\alpha) = 2 + 2\alpha$,

$\xi_3 = (3, 4)$ with $\Phi^{-1}(\alpha) = 3 + \alpha$ and $\xi_4 = (1, 2, 3)$ with $\Phi^{-1}(\alpha) = 1 + 2\alpha$. Let $a = 2$ and $b = 1$; then, this model can be formulated as follows:

$$
\begin{cases}
max\, 2k_1 + 3k_2 + 3.5k_3 + 2k_4 \\
subject\ to: \\
\int_0^1 \big(k_1(1+2\alpha) + k_2(2+2\alpha) + k_3(3+\alpha) + k_4(1+2\alpha)\big)^2 - \big(2k_1 + 3k_2 + 3.5k_3 + 2k_4\big)^2 d\alpha \le 2 \\
k_1 \int_0^1 \frac{(1+2\alpha)(2\alpha-1)}{\sqrt{1+(2\alpha-1)^2}} d\alpha + k_2 \int_0^1 \frac{(2+2\alpha)(2\alpha-1)}{\sqrt{1+(2\alpha-1)^2}} d\alpha + k_3 \int_0^1 \frac{(3+\alpha)(2\alpha-1)}{\sqrt{1+(2\alpha-1)^2}} d\alpha \\
+k_4 \int_0^1 \frac{(1+2\alpha)(2\alpha-1)}{\sqrt{1+(2\alpha-1)^2}} d\alpha \ge 1 \\
\quad k_1 + k_2 + k_3 + k_4 = 1 \\
\quad k_i \ge 0, i = 1, 2, 3, 4.
\end{cases}
$$

We further simplify this as

$$
\begin{cases}
max\, 2k_1 + 3k_2 + 3.5k_3 + 2k_4 \\
subject\ to: \\
\int_0^1 \big(k_1(1+2\alpha) + k_2(2+2\alpha) + k_3(3+\alpha) + k_4(1+2\alpha)\big)^2 d\alpha - \big(2k_1 + 3k_2 + 3.5k_3 + 2k_4\big)^2 \le 2 \\
(k_1 + k_2 + 0.5k_3 + k_4) \int_{-0.5}^{0.5} \frac{4t^2}{\sqrt{1+4t^2}} dt \ge 1 \\
\quad k_1 + k_2 + k_3 + k_4 = 1 \\
\quad k_1 \ge 0, k_2 \ge 0, k_3 \ge 0, k_4 \ge 0.
\end{cases}
$$

Then, the best portfolio selection is given as $k = (0.0901, 0.9327, 0.0000, 0.0908)$. The maximum expected value is $E[k_1\xi_1 + k_2\xi_2 + k_3\xi_3 + k_4\xi_4] = 3.1600$.

Now, a stock model of portfolios for every investor is proposed. Naturally, it can be extended and used for many kinds of portfolio selection.

8 Conclusion

In this paper, we studied the arc entropy of uncertain variables and its applications to uncertain portfolio selection and uncertain learning curves. We first proposed the concept of the arc entropy of uncertain variables via uncertainty distributions, and a formula for arc entropy was derived via inverse uncertainty distributions. It was found that for both the distributions and the inverse distributions, the entropy is the same, but this also makes the formula more scientific. We proposed a theorem of arc entropy for uncertain learning curve variables with an inverse uncertainty distribution, and its application was investigated. We also focused on arc entropy applications to portfolio selection. A portfolio selection model was established, and the mathematical method for using it was given. Importantly, two general arc entropies for uncertain learning curve variables were presented. Their applications are potential directions for future research.

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