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CMMSE: An approach to the stability of Reset Switched Systems using a dynamics without reset

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Abstract

In this paper, we consider switched systems that allow discontinuous jumps in the state when switching occurs. In particular we analyze the case where these jumps are dictated by a state reset rule. We show that it is possible to study the state trajectories of a switched system with state reset by means of some switched systems without state resets, i.e., with continuous state trajectories. We establish some sufficient conditions for stability of switched systems with state reset using the associated systems without reset. The obtained results are used in numerical examples to show the effectiveness of the proposed approach.

KEYWORDS:
Lyapunov Theory, Reset state, Switched system, Stability

1 | INTRODUCTION

A switched linear system is a special type of time-variant system that can be viewed as a family of time-invariant linear systems together with a switching law. The switching law determines which of the linear system within the family is active at each time instant, hence defining how the time-invariant systems commute among themselves, giving origin to a time-variant system,[1].

The most common approach when dealing with switched systems is not to allow jumps in the state during the switching instants. In such case, even if each individual time-invariant system is stable the correspondent switched system may be unstable,[2]. The stability of switched systems with continuous state trajectories has been widely investigated, see, for instance,[3],[4],[5] and[6].

However, in some situations it is natural and profitable to allow state jumps during switching instants. In fact, many processes may experience abrupt state changes at certain moments of time and one effective approach to achieve stability is state resetting at switching instants, see, for instance,[7] and[8]. The resulting system is a switched system with state reset,[9].

In this paper, we consider switched systems with state reset as switched systems where the state may change according with a certain linear reset map when switching occurs. This reset map does not depend on the instant when the switching occurs itself, but only on the value of the switching signal before and after the switching.

The analysis of the stability property of a switched system must take into consideration the possibility that all the components of the state or only part of them may undergo the action of a reset and, if possible, whether the choice of resets is allowed and whether there is a criterion of choice in order to ensure the stability of the switched system. These problems of stabilization of a switched system using state reset have been considered in our previous work, see, for instance,[10],[11] and[12]. Recently, other authors have considered similar stabilization problems in different frameworks, see, for instance,[13],[14] and[15].

In the present work, we consider that resets can be applied at all the switching instants, however our focus is to investigate the stability property of the switched system associated with a given class of resets. The stability analysis will be carried out from the analysis of the trajectories of a time-variant system without resets whose dynamics is derived from the dynamics of the initial system. Using the obtained results, we present some sufficient conditions for the stability of a certain particular classes of switched systems with state reset.

2 | PRELIMINARIES

Let $\Sigma_{\mathcal{P}} = \left\{\Sigma_p : p \in \mathcal{P}\right\}$ be a family of linear time-invariant systems (LTI) state-space systems, where $\mathcal{P} = \{1, \cdots, N\}$ is a finite index set. Since the focus is on internal stability, we consider systems without inputs and disregard the output equation. Let $\dot{x}(t) = A_p x(t)$ be the state-space representation of Σ_p, for $p \in \mathcal{P}$. Additionally, define a *switching law* or a *switching signal* as a piecewise constant function of time $\sigma : \left[t_0, +\infty\right[\to \mathcal{P}$, such that, $\sigma(t) = i_{k-1}$, for $t \in \left[t_{k-1}, t_k\right[_{k\in\mathbb{N}}$; the time instants t_k, $k \in \mathbb{N}$, such that $t_0 < t_1 < \ldots < t_{k-1} < t_k \ldots$ are called *switching instants*. The set of all switching signals is represented by $\mathcal{S}_{\mathcal{P}}$.

A triple $\mathbb{S} = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}\right)$ is said to be a *switched system with switching bank* $\Sigma_{\mathcal{P}}$, in the sense that each switching signal $\sigma(\cdot) \in \mathcal{S}_{\mathcal{P}}$ produces the linear time-variant system Σ_σ (called σ-switched system) defined by

$$\dot{x}(t) = A_{\sigma(t)} x(t) \tag{1}$$

for all $t \geq t_0$, where $x(t) \in \mathbb{R}^n$ is the state. With some language abuse, since LTI state-space systems are completely determined by the state matrices, we also call $\left\{A_p : p \in \mathcal{P}\right\}$ the switching bank of $\mathbb{S}$. Here, no state jumps at the switching instants t_k are allowed.

Next, we present the definition of switched system with state reset. At each switching instant the state reset is determined by the systems Σ_p that are active before and after the switching. The state discontinuities are associated with a family of reset laws

$$\mathcal{R} = \left\{R_{(q,p)} \in \mathbb{R}^{n\times n} : (q,p) \in \mathcal{P} \times \mathcal{P}, q \neq p\right\},$$

where $R_{(q,p)}$ are non-zero real matrices that act on the state of the system at the switching instants. These matrices are called *reset matrices* or, simply, *resets*.

Definition 1. A quadruple $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$ is said to be a *switched system with state reset* if each switching signal $\sigma(\cdot) \in \mathcal{S}_{\mathcal{P}}$ produces a linear time-variant system Σ_σ^R defined as in (1), such that at each switching time instant t_k, $k \in \mathbb{N}$, with $t_0 < t_1 < \ldots < t_{k-1} < t_k < \ldots$,

$$x(t_k) = R_{(i_{k-1}, i_k)} x(t_k^-), \text{ if } \sigma(t) = i_{k-1} \text{ for } t \in \left[t_{k-1}, t_k\right[, \tag{2}$$

where $x(t_k^-) := \lim\limits_{t \to t_k^-} x(t)$ and $R_{(i_{k-1}, i_k)}$ are reset matrices from $\mathcal{R}$.

Clearly, according to Definition 1, if, for all $(q,p) \in \mathcal{P} \times \mathcal{P}$, $R_{(q,p)} = I_n$, where I_n denotes the identity matrix of order n, the correspondent switched system is a system without state reset, i.e., a classical switched system where the state evolves continuously. Note that, for these systems (without reset) we use the notation $\mathbb{S} = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}\right)$, as already mentioned.

Remark 2. Let $\sigma \in \mathcal{S}_{\mathcal{P}}$ be a switching signal and $\mathcal{R}_\sigma$ be the set of reset matrices associated to the σ-switched system. Then

$$\mathcal{R}_\sigma \subseteq \mathcal{R}.$$

If $\mathcal{P} = \{1, \ldots, N\}$, then $\mathcal{R}_\sigma$ has at most $N(N-1)$ resets.

Next definition presents the notion of stability of a switched system (with or without reset) under arbitrary switching. Notice that the used notion is the usual global uniform exponential stability, which for linear systems is equivalent to the property of global uniform asymptotic stability,[1]. In this paper, we use simply the term "stable" to refer to both equivalent properties.

Definition 3. A switched system $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$ is *stable* if there exist $\gamma, \lambda \in \mathbb{R}^+$ such that, for every switching signal σ, for every $t_0 \in \mathbb{R}$ and every $x_0 \in \mathbb{R}^n$, the correspondent state trajectory $x(t)$, with $x(t_0) = x_0$, satisfies $||x(t)|| \leq \gamma e^{-\lambda(t-t_0)}||x_0||$, for $t \geq t_0$.

Clearly, this definition also covers the case of switched systems without reset, for which $\mathcal{R} = \left\{I_n\right\}$, as mentioned earlier.

Remark 4. Notice that a switched system with state reset $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$ is stable if for each switching signal $\sigma \in \mathcal{S}_{\mathcal{P}}$ the correspondent $\sigma-$switched system with state reset Σ_σ^R defined by

$$\Sigma_\sigma^R := \begin{cases} \dot{x}(t) &= A_{\sigma(t)}x(t) \\ x(t_k) &= R_{(i_{k-1},i_k)}x(t_k^-) \end{cases}, \quad R_{(i_{k-1},i_k)} \in \mathcal{R}_\sigma, \tag{3}$$

is stable.

For a switched system to be stable, it is necessary to have a bank of stable invariant systems. So, in the following, we exclusively consider switched systems that have a switching bank $\Sigma_{\mathcal{P}}$ formed by stable systems. On the other hand, a well-known condition that guarantees the stability of a switched system $\mathbb{S}$ is based on the notion of common quadratic Lyapunov function (CQLF). A function $V(x) = x^T Px$, where P is a square symmetric positive definite matrix, is said to be a CQLF for a switched system $\mathbb{S}$ if

$$A_p^T P + PA_p < 0, \text{ for all } p \in \mathcal{P}.$$

With some abuse of language, we shall call the matrix P a CQLF for $\Sigma_{\mathcal{P}}$ and for A_p, $p \in \mathcal{P}$. The existence of a CQLF is sufficient to ensure stability of the switched system, under arbitrary switching signal as is stated in the following proposition.

Proposition 5. [4] The switched system $S = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}\right)$ is stable if there exists a CQLF for $\Sigma_{\mathcal{P}}$.

Among the results related with the existence of a CQLF for a switched system, see, for instance,[5],[16] and[17], we present the following propositions in order to improve the readability of Section 4 of this paper.

Proposition 6. [18] Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be stable matrices. A_1 and A_2 have CQLF if and only if A_1A_2 and $A_1A_2^{-1}$ do not have negative real eigenvalues.

Proposition 7. [19] Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be stable matrices such that $\operatorname{rank}(A_1 - A_2) = 1$. The matrices A_1 and A_2 have CQLF if and only if A_1A_2 does not have negative real eigenvalues.

3 | REDUCTION TO A DYNAMICS WITHOUT RESET

In this section, we analyze the stability property of a switched system with state reset $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$ analyzing the property of each σ-switched system with state reset Σ_σ^R. This is carried out by relating the state trajectories of the $\mathbb{S}_R$ with the ones of a time-variant system without resets, i.e., with no jumps on the state.

We next produce a switching dynamics without reset that allows an analysis of the trajectories of the reset switched system Σ_σ^R. The trajectories of this new dynamics will be denoted by $\widetilde{x}(t)$. Here, to simplify the notation we define $c_k = (i_{k-1}, i_k)$, for $k \in \mathbb{N}$.

Lemma 8. Let Σ_σ^R be a σ-switched system with state reset defined according to (3). Assume that all the reset matrices are invertible. The following time-variant linear system

$$\dot{\widetilde{x}}(t) = \widetilde{A}(t)\widetilde{x}(t) \tag{4}$$

where

$$\begin{aligned} \widetilde{A}(t) &= A_{i_0}, \text{ para } t \in \left[t_0, t_1\right[; \\ \widetilde{A}(t) &= \left(\prod_{m=k}^{1} R_{c_m}\right)^{-1} A_{i_k} \prod_{m=k}^{1} R_{c_m}, \text{ for } t \in \left[t_k, t_{k+1}\right[, \ k \in \mathbb{N}; \end{aligned} \tag{5}$$

and $\widetilde{x}(t_k) = \widetilde{x}(t_k^-)$ is such that

$$x(t) = \prod_{m=k}^{0} R_{c_m}\widetilde{x}(t), \text{ for } t \in \left[t_k, t_{k+1}\right[, \ k \in \mathbb{N}_0.$$

By convention, $R_{c_0} = I_n$.

Proof. For $t \in \left[t_0, t_1\right[$, $\dot{x}(t) = A_{i_0}x(t)$, that is equivalent to $\dot{\widetilde{x}}_0(t) = \widetilde{A}(t)\widetilde{x}_0(t)$, where $\widetilde{x}_0(t_0) = x(t_0)$, such that

$$\widetilde{x}_0(t) = x(t) \text{ and } \widetilde{A}(t) = A_{i_0}. \tag{6}$$

For $t \in \left[t_1, t_2\right[$, $\dot{x}(t) = A_{i_1}x(t)$, with $x(t_1) = R_{c_1}x(t_1^-)$. So

$$R_{c_1}^{-1}\dot{x}(t) = R_{c_1}^{-1}A_{i_1}R_{c_1}R_{c_1}^{-1}x(t)$$

and $R_{c_1}^{-1}x(t_1) = x(t_1^-)$, which, by (6), this is equivalent to $x(t_1^-) = \widetilde{x}_0(t_1^-)$. Consequently,

$$R_{c_1}^{-1}\dot{x}(t) = R_{c_1}^{-1}A_{i_1}R_{c_1}R_{c_1}^{-1}x(t) \;\text{ and }\; R_{c_1}^{-1}x(t_1) = \widetilde{x}_0(t_1^-). \tag{7}$$

Taking

$$\widetilde{x}_1(t) = R_{c_1}^{-1}x(t), \tag{8}$$

and considering (7), we obtain

$$\dot{\widetilde{x}}_1(t) = \widetilde{A}(t)\widetilde{x}_1(t) \;\text{ and }\; \widetilde{x}_1(t_1) = \widetilde{x}_0(t_1^-),$$

for $t \in \left[t_1, t_2\right[$ and $\widetilde{A}(t) = R_{c_1}^{-1}A_{i_1}R_{c_1}$.

For $t \in \left[t_2, t_3\right[$, $\dot{x}(t) = A_{i_2}x(t)$, with $x(t_2) = R_{c_2}x(t_2^-)$. Hence, by (8), we have $R_{c_1}\widetilde{x}_1(t_2^-) = x(t_2^-)$ and consequently

$$x(t_2) = R_{c_2}R_{c_1}\widetilde{x}_1(t_2^-).$$

Therefore,

$$R_{c_1}^{-1}R_{c_2}^{-1}\dot{x}(t) = R_{c_1}^{-1}R_{c_2}^{-1}A_{i_2}R_{c_2}R_{c_1}R_{c_1}^{-1}R_{c_2}^{-1}x(t) \;\text{ and }\; R_{c_1}^{-1}R_{c_2}^{-1}x(t_2) = \widetilde{x}_1(t_2^-). \tag{9}$$

Taking

$$\widetilde{x}_2(t) = R_{c_1}^{-1}R_{c_2}^{-1}x(t),$$

(9) is equivalent to

$$\dot{\widetilde{x}}_2(t) = \widetilde{A}(t)\widetilde{x}_2(t); \;\; \widetilde{x}_2(t_2) = \widetilde{x}_1(t_2^-),$$

for $t \in \left[t_2, t_3\right[$ and $\widetilde{A}(t) = R_{c_1}^{-1}R_{c_2}^{-1}A_{i_2}R_{c_2}R_{c_1}$.

Following the previous process, we obtain for $t \in \left[t_k, t_{k+1}\right[$

$$\dot{\widetilde{x}}_k(t) = \widetilde{A}(t)\widetilde{x}_k(t), \tag{10}$$

where

$$\begin{aligned} \widetilde{x}_k(t_k) &= \tilde{x}_{k-1}(t_k^-) \\ \widetilde{A}(t) &= \left(\prod_{m=1}^{k} R_{c_m}^{-1}\right) A_{i_k} \prod_{m=k}^{1} R_{c_m} \\ \widetilde{x}_k(t) &= \left(\prod_{m=1}^{k} R_{c_m}^{-1}\right) x(t), \end{aligned} \tag{11}$$

with $k \in \mathbb{N}$.

Considering $\widetilde{x}(t) = \widetilde{x}_k(t)$, for $t \in \left[t_k, t_{k+1}\right[$, $k \in \mathbb{N}_0$, (10) and (11) can be written as

$$\dot{\widetilde{x}}(t) = \widetilde{A}(t)\widetilde{x}(t); \tag{12}$$

where

$$\begin{aligned} \widetilde{A}(t) &= A_{i_0}, \text{ for } t \in \left[t_0, t_1\right[\\ \widetilde{A}(t) &= \left(\prod_{m=k}^{1} R_{c_m}\right)^{-1} A_{i_k} \prod_{m=k}^{1} R_{c_m}, \text{ para } t \in \left[t_k, t_{k+1}\right[, \; k \in \mathbb{N}, \end{aligned}$$

and $\widetilde{x}(t_k) = \tilde{x}(t_k^-)$. From (11), it follows that

$$x(t) = \prod_{m=k}^{0} R_{c_m}\widetilde{x}(t), \text{ for } t \in \left[t_k, t_{k+1}\right[, \; k \in \mathbb{N}_0.$$

□

In order to avoid long mathematical expressions, we will use the following notation:

$$\begin{aligned} \overline{R}_{\sigma,0} &:= I_n \\ \overline{R}_{\sigma,k} &:= \textstyle\prod_{m=k}^{1} R_{c_m}, k \in \mathbb{R}. \end{aligned} \tag{13}$$

Remark 9. Notice that, in the proof of Lemma (8), we have associated to the original σ-switched system with state reset a time-variant system with a linear and piecewise invariant dynamics. That dynamics is determined by the following set of stable matrices:

$$\widetilde{\mathcal{A}} = \left\{ A_{i_0}, \overline{R}_{\sigma,1}^{-1} A_{i_1} \overline{R}_{\sigma,1}, \overline{R}_{\sigma,2}^{-1} A_{i_2} \overline{R}_{\sigma,2}, \dots \right\}.$$

Although the number of distinct reset matrices is finite, the set $\widetilde{\mathcal{A}}$ may be infinite. In this case, the time-variant system (4) does not fit into the definition of σ-switched system without reset, (1). Nevertheless, if

$$\prod_{\sigma} := \left\{ \overline{R}_{\sigma,k} \,:\, t_k \text{ is a switching instant for } \sigma \right\},$$

is finite, then the time-variant linear system (4) can be considered a $\widetilde{\sigma}$-switched system (without reset) for some switching signal $\widetilde{\sigma}$ that supervises the switching among those systems. This switching signal, $\widetilde{\sigma}$, will have switching instants in the same set of switching instants of σ. Indeed, let us suppose that

$$\prod_{\sigma} = \left\{ \widehat{R}_l \,:\, l \in \mathcal{L} = \{0, \dots, T\} \right\}, \text{ for some } T \in \mathbb{N}.$$

Then $\widetilde{x}(t)$ corresponds to a trajectory of a switched system for which the switching bank can be associated to the set

$$\widetilde{\mathcal{A}} = \left\{ \widetilde{A}_{(p,l)} = \widehat{R}_l^{-1} A_p \widehat{R}_l \,:\, (p,l) \in \mathcal{Q} \right\}$$

for a given switching signal $\widetilde{\sigma} \,:\, \left[t_0, +\infty\right[\to \mathcal{Q}$, where $\mathcal{Q} \subset \mathcal{P} \times \mathcal{L}$. Moreover, $\widetilde{\sigma}$ will have switching instants in the same set of switching instants of σ.

The next result assures that, for a switching signal σ and under certain conditions, if the time-variant system (4) is stable, then the σ- switched system with state reset Σ_σ^R is stable too. We denote by $\|\cdot\|$ the spectral norm (i.e., the maximum singular value of a matrix).

Theorem 10. Let Σ_σ^R be a σ-switched system with state reset, defined as in (3), for which $\left\{ \left\|\overline{R}_{\sigma,k}\right\| \,:\, k \in \mathbb{N} \right\}$ is upper bounded. If the time-variant system (4) is stable, then Σ_σ^R is stable.

Proof. Let Σ_σ^R be a σ-switched system with state reset defined in (3) associated to a switching signal $\sigma \,:\, \left[t_0, +\infty\right[\longrightarrow \mathcal{P}$, with switching instants $t_1 < \dots < t_k < \dots, k \in \mathbb{N}$.

By Lemma 8,

$$\widetilde{x}_k(t) = \widetilde{x}(t) \;\text{ and }\; \sigma(t_k) = \sigma(t), \;\text{ for any } t > t_0,$$

where $\widetilde{x}(t)$ is the trajectory of the system (12), i.e., $\dot{\widetilde{x}}(t) = \widetilde{A}(t)\widetilde{x}(t)$. Then, for any $t > t_0$, there exists $k \in \mathbb{N}$ such that

$$\widetilde{x}(t) = \overline{R}_{\sigma,k}^{-1} x(t).$$

Therefore for each $t > 0$,

$$\|x(t)\| \leq \left\|\overline{R}_{\sigma,k}\right\| \|\widetilde{x}(t)\|, \text{ for some } k \in \mathbb{N}.$$

Since $\left\{ \left\|\overline{R}_{\sigma,k}\right\| \,:\, k \in \mathbb{N} \right\}$ is an upper bounded set, then there exists $L > 0$ such that $\left\|\overline{R}_{\sigma,k}\right\| < L$. Consequently, $\|x(t)\| \leq L\|\widetilde{x}(t)\|$ and the σ-switched system with state reset Σ_σ^R is stable. □

Remark 11. Notice that, additionally, if the set $\left\{ \left\|\overline{R}_{\sigma,k}\right\|^{-1} \,:\, k \in \mathbb{N} \right\}$ is upper bounded, then the sufficient condition of the theorem is also necessary. For example, this happens when the set of matrices $\left\{ \overline{R}_{\sigma,k} \,:\, k \in \mathbb{N} \right\}$ is finite.

Remark 12. For some families of resets, the set $\left\{ \left\|\overline{R}_{\sigma,k}\right\| \,:\, k \in \mathbb{N} \right\}$ is upper bounded, for every switching signal.

One of those cases is analyzed in the next section.

4 | STABILITY ANALYSIS UNDER RESTRICTED RESETTING

In this section, we consider that the set of reset matrices is restricted to have a certain structure and briefly illustrate how to apply Theorem 10 in the stability analysis of a switched system with state reset $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$, with $\mathcal{P} = \{1, 2\}$. More precisely, we consider that reset matrices $R_{(q,p)}$, with $q, p \in \mathcal{P}$, are such that:

$$R_{(q,p)} = \begin{cases} R, & \text{if } p = 1 \\ R^{-1}, & \text{if } p = 2 \end{cases},$$

where $q, p \in \mathcal{P}$ are the values that σ takes at the switching instants t_{k-1} and t_k, $k \in \mathbb{N}$, respectively. Thus $R_{(2,1)} = R$ and $R_{(1,2)} = R^{-1}$.

In light of the results established in the previous section, namely Lemma (8) and Theorem (10), stability analysis of $\mathbb{S}_R$ can be done analyzing the stability property of each $\sigma-$switched system with state reset as is described next, in (i) and (ii).

(i) For each switching signal σ such that $\sigma(t) = 1$ for $t \in \left[t_0, t_1\right[$, taking into account (13), we have

$$\begin{aligned} \overline{R}_{\sigma,0} &= I_n, \\ \overline{R}_{\sigma,1} &= R_{(1,2)} = R^{-1}, \\ \overline{R}_{\sigma,2} &= R_{(2,1)} R_{(1,2)} = R R^{-1} = I_n \end{aligned}$$

and so on, i.e., we obtain the finite set

$$\left\{ \overline{R}_{\sigma,k} : k \in \mathbb{N}_0 \right\} = \left\{ I, R^{-1} \right\}.$$

Consequently, the obtained time-variant system is a switched system, say σ_1-switched system (without reset), for the switching signal $\sigma_1 : \left[t_0, +\infty\right[\to \mathcal{P}$ such that

$$\sigma_1(t) = \begin{cases} 1 & \text{if } t \in \left[t_{2k-2}, t_{2k-1}\right[\\ 2 & \text{if } t \in \left[t_{2k-1}, t_{2k}\right[\end{cases}, k \in \mathbb{N},$$

with the following (finite) switching bank

$$\widetilde{\mathcal{A}}_1 = \left\{ A_{i_0}, \overline{R}_{\sigma,1}^{-1} A_{i_1} \overline{R}_{\sigma,1}, \overline{R}_{\sigma,2}^{-1} A_{i_2} \overline{R}_{\sigma,2}, \dots \right\} = \left\{ A_1, R A_2 R^{-1} \right\}.$$

Moreover, being $\left\{ \left\| \overline{R}_{\sigma,k} \right\| : k \in \mathbb{N} \right\}$ upper bounded, according to Theorem 10, if the σ_1-switched system (without reset) is stable, then the associated reset σ_1-switched system $\Sigma^R_{\sigma_1}$ is stable.

(ii) Similarly, for each switching signal σ such that $\sigma(t) = 2$ for $t \in \left[t_0, t_1\right[$, we obtain a σ_2-switched system (without resets), for the switching signal $\sigma_2 : \left[t_0, +\infty\right[\to \mathcal{P}$, such that

$$\sigma_2(t) = \begin{cases} 2 & \text{if } t \in \left[t_{2k-2}, t_{2k-1}\right[\\ 1 & \text{if } t \in \left[t_{2k-1}, t_{2k}\right[\end{cases}, k \in \mathbb{N},$$

with the following finite switching bank

$$\widetilde{\mathcal{A}}_2 = \left\{ A_2, R^{-1} A_1 R \right\}.$$

According to Theorem 10, if the σ_2-switched system (without reset) is stable, then the associated reset σ_2-switched system $\Sigma^R_{\sigma_2}$ is stable.

These conclusions allow to state the following proposition.

Proposition 13. Let $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$ be a switched system with state reset, where $\mathcal{P} = \{1, 2\}$, $\Sigma_{\mathcal{P}}$ is the switching bank associated to the matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$ and $\mathcal{R}$ is the set of reset matrices $R_{(q,p)}$ defined as

$$R_{(q,p)} = \begin{cases} R, & \text{if } p = 1 \\ R^{-1}, & \text{if } p = 2 \end{cases}, \tag{14}$$

being q and p the values that σ takes at the switching instants t_{k-1} and t_k, $k \in \mathbb{N}$.
The system $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$ is stable if the switched systems $\left(\mathcal{P}, \Sigma^1_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}\right)$ and $\left(\mathcal{P}, \Sigma^2_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}\right)$, with switching banks $\Sigma^1_{\mathcal{P}}$ and $\Sigma^2_{\mathcal{P}}$ associated to $\left\{ A_1, R A_2 R^{-1} \right\}$ and $\left\{ A_2, R^{-1} A_1 R \right\}$, respectively, are stable.

Using a CQLF approach the next two corollaries are established, using Propositions (6) and (7).

Corollary 14. Let $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$ be a switched system with state reset, where $\mathcal{P} = \{1, 2\}$, $\Sigma_{\mathcal{P}}$ is the the switching bank associated to the matrices $A_1, A_2 \in \mathbb{R}^{2\times 2}$ and $\mathcal{R}$ is a set of reset matrices $R_{(q,p)}$ defined as in (14).
The system $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$ is stable if $A_1RA_2R^{-1}$ and $A_1RA_2^{-1}R^{-1}$ do not have real negative eigenvalues.

Proof. If $A_1RA_2R^{-1}$ and $A_1RA_2^{-1}R^{-1}$ do not have real negative eigenvalues, then A_1 and RA_2R^{-1} have CQLF (by Proposition (6)). On the other hand, it is easy to see that $A_1RA_2R^{-1}$ and $A_1RA_2^{-1}R^{-1}$ do not have real negative eigenvalues if and only if $R^{-1}A_1RA_2$ and $R^{-1}A_1RA_2^{-1}$ do not have real negative eigenvalues. So, A_2 and $R^{-1}A_1R$ have also a CQLF. Therefore, the switched systems $\left(\mathcal{P}, \Sigma^1_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}\right)$ and $\left(\mathcal{P}, \Sigma^2_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}\right)$ are stable and the switched system with state reset $\mathbb{S}_R$ is stable. □

Corollary 15. Let $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$ be a switched system with state reset, where $\mathcal{P} = \{1, 2\}$, $\Sigma_{\mathcal{P}}$ is the the switching bank associated to the matrices $A_1, A_2 \in \mathbb{R}^{n\times n}, n \in \mathbb{N}$, such that $\text{rank}(A_1 - A_2) = 1$, and $\mathcal{R}$ is a set of reset matrices $R_{(q,p)}$ defined as in (14).
The system $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$ is stable if $A_1RA_2R^{-1}$ does not have real negative eigenvalues.

Example 16. The switched system with state reset $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$, as in Proposition 13, where

$$A_1 = \begin{bmatrix} -0.05 & 2 \\ -1 & -0.05 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.05 & 1 \\ -2 & -0.05 \end{bmatrix} \text{ and } R_{(2,1)} = R_{(1,2)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is stable, since, by Corollary 14, the matrices

$$\widetilde{A}_1 = A_1 \quad \text{and} \quad \widetilde{A}_2 = RA_2R^{-1} = \begin{bmatrix} -0.05 & -2 \\ 1 & -0.05 \end{bmatrix}$$

are such that $\widetilde{A}_1\widetilde{A}_2$ and $\widetilde{A}_1\widetilde{A}_2^{-1}$ do not have real negative eigenvalues.

Example 17. The switched system with state reset $\mathbb{S}_R = \left(\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R}\right)$, as in Proposition 13, where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -15 & -2 \end{bmatrix}$$

and

$$R_{(2,1)} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } R_{(1,2)} = \begin{bmatrix} 0 & 0.5 & 0 \\ 1 & -0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

is stable, since, by Corollary 15, the matrices

$$\widetilde{A}_1 = A_1 \quad \text{and} \quad \widetilde{A}_2 = RA_2R^{-1} = \begin{bmatrix} 1 & -0.5 & 1 \\ 2 & -1 & 0 \\ -15 & -2.5 & -2 \end{bmatrix}$$

are such that $\widetilde{A}_1\widetilde{A}_2$ does not have real negative eigenvalues.

5 | CONCLUSION

In this paper we have investigated the stability property of a switched system in which, once switching occurs, the state is forced to assume (is reset to) a new value which is a linear function of the previous state. This type of system is called a switched system with state reset. We were able to identify the relation between the trajectories of a switched system with state reset and the trajectories of a time-variant system (without reset). This reduction to a dynamics with continuous state trajectories, presented in Lemma 8, has allowed a different approach to the study of the stability of switched systems with state reset. It turns out that, for bounded sets of resets, the stability of the associated time-variant system (without reset) implies the stability of the original system (with reset), Theorem 10. Additionally, we have applied this result in the study of systems that have a particular type of resetting. In this case, the sets of resets are bounded, which allowed to establish two sufficient conditions for stability. Two illustrative examples were included. In future research, we expect to identify other classes of systems where this approach may be carried out with identical success.

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Conflict of interest

The authors declare no potential conflict of interests.

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