

The backward problem for radially symmetric time-fractional diffusion-wave equation under Robin boundary condition

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Abstract

This paper is devoted to solve the backward problem for radially symmetric time-fractional diffusion-wave equation under Robin boundary condition. This problem is ill-posed and we apply an iterative regularization method to solve it. The error estimates are obtained under the a priori and a posteriori parameter choice rules. Numerical results show that the proposed method is efficient and stable.

Keywords: backward problem; radially symmetric time-fractional diffusion-wave equation; iterative regularization method; error estimates

1. Introduction

Time-fractional diffusion-wave equation plays an important role in many scientific and engineering fields, such as biochemistry, biological system, physics, finance, geology and ecology. Since the pioneering paper[1], the work on time-fractional diffusion-wave equation mushroomed. The direct problems for time-fractional diffusion-wave equation have been investigated extensively[2, 3]. And the inverse problems connected with time-fractional diffusion-wave equation have some papers such as inverse source problems[4, 5], inverse initial value problems[6, 7], inverse coefficient problems [8].

In this paper, we are interested in studying the backward problems of the radially symmetric time-fractional diffusion-wave equation. There are only a few papers [9, 10, 11] on the inverse problem of the radially symmetric time-fractional diffusion or diffusion-wave equation, but these papers are limited to the Dirichlet boundary condition (1.1) or the Neumann boundary condition(1.2).

$$u(r, t) = \sigma(t), \quad (r, t) \in \partial D \times [0, T]. \quad (1.1)$$

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$$\frac{\partial u}{\partial v} = \zeta(t) \quad (r, t) \in \partial D \times [0, T]. \quad (1.2)$$

A more general boundary condition is the Robin boundary condition (1.3).

$$\frac{\partial u}{\partial v} + \beta u = \mu(t) \quad (r, t) \in \partial D \times [0, T], \quad (1.3)$$

where β is the boundary heat transfer coefficient, also known as the Robin coefficient. Under certain conditions, the Robin boundary condition can be transformed into a Dirichlet boundary condition or a Neumann boundary condition [12]. When $\beta = \infty$, the limiting form of equation (1.3) is the Dirichlet boundary condition (1.1), and when $\beta = 0$, equation (1.3) is the Neumann boundary condition (1.2). Robin boundary condition describes the relationship between heat flow exchange and ambient temperature at the boundary condition, which is more general and in line with the actual situation, and is also more complicated and more meaningful to study compared to Dirichlet and Neumann boundary conditions. However, to our knowledge, there are almost no papers on the inverse problem of the radially symmetric time-fractional diffusion-wave equation under robin boundary condition. Motivated by Povstenko [13], we focus on the backward problem of the radially symmetric time-fractional diffusion-wave equation under Robin boundary condition. And we will use an iterative regularization method to solve this inverse problem. This iterative regularization method is more general than the iterative regularization method in [14] and also establishes a connection with the classical Landweber iterative [15] and modified fractional Landweber iterative [9] methods.

We consider the following radially symmetric time-fractional diffusion-wave equation with the source term

$$D_t^\alpha u(r, t) = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} [r^{d-1} \frac{\partial u(r, t)}{\partial r}] + f(r, t), \quad (1.4)$$

with the initial conditions

$$t = 0 : u = \phi(r), \quad 0 \leq r \leq R, \quad (1.5)$$

$$t = 0 : \frac{\partial u}{\partial t} = \psi(r), \quad 0 \leq r \leq R, \quad (1.6)$$

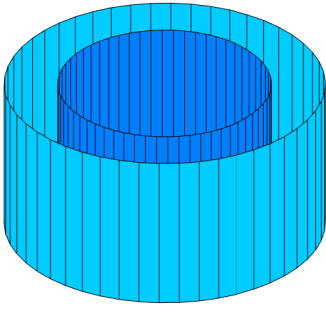
and the Robin boundary condition

$$r = R : \frac{\partial u}{\partial r} + \beta u = \mu(t), \quad 0 \leq t \leq T, \quad (1.7)$$

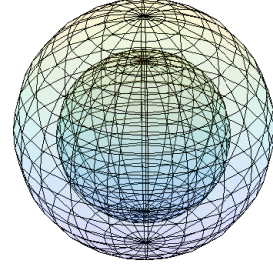
where β is the Robin coefficient, $D_t^\alpha u(r, t)$ represents the Caputo fractional derivative of order α ($1 < \alpha < 2$) defined by

$$D_t^\alpha u(r, t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \frac{\partial^2 u(r, \tau)}{\partial \tau^2} d\tau, \quad 1 < \alpha < 2,$$

and $d \in \{2, 3\}$. When $d = 2$, the equation (1.4) is the time-fractional diffusion-wave equation in a cylinder and when $d = 3$, the equation (1.4) is the time-fractional diffusion-wave equation in a sphere. These two equations have important applications in industrial and chemical fields, such as cylindrical steel tanks for blast furnace steelmaking, cylindrical reactors for catalyst production, spherical vessels for liquefied gases transportation, etc.



(a) *Cylinder model*



(b) *Sphere model*

Figure 1: The two radially symmetric models.

If $f(r, t)$, $\phi(r)$, $\psi(r)$ and $\mu(t)$ are known, problem (1.4)-(1.7) is a classic direct problem. The inverse problem here is to reconstruct the initial value $\phi(r)$ according to the additional condition

$$u(r, T) = g(r), \quad 0 \leq r \leq R. \quad (1.8)$$

Since the measurement is noise-contaminated inevitably, we assume that $g^\delta(r)$ is the noisy measurement of $g(r)$ which satisfies

$$\|g^\delta(r) - g(r)\| \leq \delta, \quad (1.9)$$

where $\|\cdot\|$ denotes $L^2([0, R], r)$ norm and $\delta > 0$ is the known noise level. In this paper, we mainly analyze the case of $d = 2$, while giving the corresponding remark for the case of $d = 3$.

This paper is organized as follows. In Section 2, we give some preliminary material, present the formulation of the problem, and analysis the ill-posedness of this backward problem. In Section 3, we introduce an iterative regularization method and provide the error estimates under two parameter choice rules. Numerical results are shown in Section 4. Finally, we give a conclusion in Section 5.

2. Formulation for the problem and ill-posed inverse problem

We introduce the Lebesgue space associated with the measure rdr , i.e.

$$L^2([0, R], r) = \{v : \Omega \rightarrow \mathbb{R} \text{ measurable}; \int_0^R v(r)rdr < +\infty\},$$

which is a Hilbert space with the inner product

$$(u, v) = \int_0^R u(r)v(r)rdr,$$

and the corresponding norm is defined by

$$\|v\|_{L^2([0, R], r)} = \left(\int_0^R v^2(r)rdr \right)^{\frac{1}{2}}.$$

Definition 2.1 [16]: The Mittag-Leffler function is defined by

$$E_{\alpha, \eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \eta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants.

Definition 2.2 [17]:

$$H_p = \{v \in L^2([0, R], r) : \sum_{n=1}^{\infty} \lambda_n^p (v, \omega_n)^2\}, \quad p > 0,$$

where (\cdot, \cdot) is the inner product of $L^2([0, R], r)$, H_p is a Hilbert space, and the norm is defined by

$$\|v\|_{H_p} = \left(\sum_{n=1}^{\infty} \lambda_n^p (v, \omega_n)^2 \right)^{\frac{1}{2}}.$$

Lemma 2.1 [18]: If $1 < \alpha < 2$, $\beta \in \mathbb{R}$, $\eta > 0$, there holds

$$E_{\alpha, \beta}(-\eta) = \frac{1}{\Gamma(\beta - \alpha)\eta} + \frac{1}{O(\eta^2)}, \quad \eta \rightarrow \infty.$$

Lemma 2.2 [17]: For $1 < \alpha < 2$ and any fixed $T > 0$, there exists at most finite points such that $E_{\alpha, 1}(-\lambda_n T^\alpha) = 0$. Denote the point set which makes $E_{\alpha, 1}(-\lambda_n T^\alpha) = 0$ is $I_1 = \{m_1, m_2, \dots, m_j\}$.

Lemma 2.3 [17]: For $1 < \alpha < 2$ and λ_n satisfying $\lambda_n \geq \lambda_1 > 0$, there exist positive constants \underline{C} and \overline{C} depending on α, T and λ_1 , such that

$$\frac{\underline{C}}{\lambda_n} \leq |E_{\alpha, 1}(-\lambda_n T^\alpha)| \leq \frac{\overline{C}}{\lambda_n}, \quad n \notin I_1.$$

Lemma 2.4 [19]: For $0 < y < 1$ and $k \geq 1$, define $p_k(y) = \sum_{i=0}^{k-1} (1-y)^i$ and $r_k(y) = 1 - yp_k(y) = (1-y)^k$. Then

$$\begin{aligned} p_k(y)y^\mu &\leq k^{1-\mu}, & 0 \leq \mu \leq 1, \\ r_k(y)y^\nu &\leq \theta_\nu(k+1)^{-\nu}, \end{aligned}$$

where

$$\theta_\nu = \begin{cases} 1, & 0 \leq \nu \leq 1, \\ \nu^\nu, & \nu > 1. \end{cases}$$

After introducing the definitions and lemmas that will be used in this paper, we first consider solving the direct problem. Using the method of separation of variables and Laplace transform of Mittag-Leffler function, we can obtain the solution of the direct problem (1.4)-(1.7) for $d = 2$. Now let us describe the process of solving the direct problem in more details.

We assume

$$u(r, t) = W(r, t) + V(r, t),$$

where

$$V(r, t) = \frac{\mu(t)}{R\beta + 1}r,$$

and $W(r, t)$ is the solution of the following problem with homogeneous boundary conditions:

$$\begin{cases} D_t^\alpha W(r, t) = \frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} + \tilde{f}(r, t), & 0 < r < R, 0 < t < T, \\ W(r, 0) = \tilde{\phi}(r), & 0 \leq r \leq R, \\ W_t(r, 0) = \tilde{\psi}(r), & 0 \leq r \leq R, \\ W_r(R, t) + \beta W(R, t) = 0, & 0 \leq t \leq T, \end{cases} \quad (2.1)$$

with

$$\begin{aligned} \tilde{f}(r, t) &= f(r, t) - \frac{\partial^\alpha V}{\partial t^\alpha} + \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r}, \\ \tilde{\phi}(r) &= \phi(r) - \frac{\mu(0)}{R\beta + 1}r, \\ \tilde{\psi}(r) &= \psi(r) - \frac{\mu'(0)}{R\beta + 1}r. \end{aligned}$$

Let $W(r, t) = x(r)y(t)$, substitute it into the corresponding homogeneous equation for (2.1), then we get

$$\begin{cases} x''(r) + \frac{1}{r}x'(r) + \lambda x(r) = 0, \\ x'(R) + \beta x(R) = 0. \end{cases}$$

This problem has the eigenvalues

$$\lambda_n = \left(\frac{\xi_n}{R}\right)^2, \quad n = 1, 2, 3, \dots,$$

and the corresponding eigenfunctions

$$x_n(r) = J_0\left(\frac{\xi_n}{R}r\right), \quad n = 1, 2, 3, \dots,$$

where $\{\xi_n\}_{n=1}^{\infty}$ are the positive roots of the transcendental equation [13]

$$\xi_n J_1(\xi_n) = R\beta J_0(\xi_n).$$

Then the standard orthogonal basis functions in $L^2([0, R], r)$ can be expressed as

$$\omega_n(r) = \frac{\sqrt{2}\xi_n}{\sqrt{R^2\beta^2 + \xi_n^2}R J_0(\xi_n)} J_0\left(\frac{\xi_n}{R}r\right), \quad n = 1, 2, 3, \dots$$

Further, the solution $W(r, t)$ and source term $\tilde{f}(r, t)$ of (2.1) can be expressed as

$$W(r, t) = \sum_{n=1}^{\infty} y_n(t) \omega_n(r), \tag{2.2}$$

$$\tilde{f}(r, t) = \sum_{n=1}^{\infty} \tilde{f}_n(t) \omega_n(r), \tag{2.3}$$

where $\tilde{f}_n(t) = (\tilde{f}(r, t), \omega_n(r))$. By substituting (2.2) and (2.3) into (2.1), the equation reduces to

$$\begin{cases} D_t^\alpha y_n(t) + \lambda_n y_n(t) = \tilde{f}_n(t), \\ y_n(0) = \tilde{\phi}_n, \\ y'_n(0) = \tilde{\psi}_n, \end{cases} \tag{2.4}$$

where $\tilde{\phi}_n = (\tilde{\phi}(r), \omega_n(r))$ and $\tilde{\psi}_n = (\tilde{\psi}(r), \omega_n(r))$. Using Laplace transform, we can obtain the exact solution of the problem (2.4) as follows

$$y_n(t) = \tilde{\phi}_n E_{\alpha,1}(-\lambda_n t^\alpha) + \tilde{\psi}_n t E_{\alpha,2}(-\lambda_n t^\alpha) + \int_0^t \tilde{f}_n(\tau) (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) d\tau. \tag{2.5}$$

By substituting (2.5) into (2.2), we get

$$W(r, t) = \sum_{n=1}^{\infty} [\tilde{\phi}_n E_{\alpha,1}(-\lambda_n t^\alpha) + \tilde{\psi}_n t E_{\alpha,2}(-\lambda_n t^\alpha) + \int_0^t \tilde{f}_n(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) d\tau] \omega_n(r).$$

Therefore, the solution of the problem (1.4)-(1,7) is

$$u(r, t) = \sum_{n=1}^{\infty} [\tilde{\phi}_n E_{\alpha,1}(-\lambda_n t^\alpha) + \tilde{\psi}_n t E_{\alpha,2}(-\lambda_n t^\alpha) + \int_0^t \tilde{f}_n(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) d\tau] \omega_n(r) + \frac{\mu(t)}{R\beta+1} r. \quad (2.6)$$

Remark 2.1: For the case $d = 3$, the solution has the same form as (2.6)

$$u(r, t) = \sum_{n=1}^{\infty} [\tilde{\phi}_n E_{\alpha,1}(-\lambda_n t^\alpha) + \tilde{\psi}_n t E_{\alpha,2}(-\lambda_n t^\alpha) + \int_0^t \tilde{f}_n(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) d\tau] \omega_n(r) + \frac{\mu(t)}{R\beta+1} r,$$

with the difference being the eigenvalues and eigenfunctions. The eigenvalues and eigenfunctions have the following form

$$\lambda_n = \left(\frac{\xi_n}{R}\right)^2, \quad x_n(r) = \frac{\sin\left(\frac{\xi_n}{R}r\right)}{\frac{\xi_n}{R}r}, \quad n = 1, 2, 3, \dots,$$

where $\{\xi_n\}_{n=1}^{\infty}$ are the positive roots of the transcendental equation

$$\xi_n \cos(\xi_n) = (1 - R\beta) \sin(\xi_n),$$

and the corresponding standard orthogonal basis functions in $L^2([0, R], r^2)$ can be expressed as

$$\omega_n(r) = \sqrt{\frac{2\xi_n^2}{R^3(1 - \frac{\sin(2\xi_n)}{2\xi_n})}} \frac{\sin\left(\frac{\xi_n}{R}r\right)}{\frac{\xi_n}{R}r}, \quad n = 1, 2, 3, \dots$$

According to (2.6), let $v^0(r) = \frac{\mu(0)}{R\beta+1}r$, $v^T(r) = \frac{\mu(T)}{R\beta+1}r$. Applying the final value data $u(r, T) = g(r)$, we have

$$g_n = \tilde{\phi}_n E_{\alpha,1}(-\lambda_n T^\alpha) + \tilde{\psi}_n T E_{\alpha,2}(-\lambda_n T^\alpha) + \int_0^T \tilde{f}_n(\tau)(T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha) d\tau + v_n^T,$$

where $g_n = (g(r), \omega_n(r))$, $v_n^T = (v^T(r), \omega_n(r))$. Note that $\tilde{\phi}_n = \phi_n - v_n^0$, where $\phi_n = (\phi(r), \omega_n(r))$, $v_n^0 = (v^0(r), \omega_n(r))$, we can get

$$\phi_n = \frac{h_n}{E_{\alpha,1}(-\lambda_n T^\alpha)},$$

where

$$\begin{aligned} h_n &= g_n + v_n^0 E_{\alpha,1}(-\lambda_n T^\alpha) - \tilde{\psi}_n T E_{\alpha,2}(-\lambda_n T^\alpha) \\ &\quad - \int_0^T \tilde{f}_n(\tau) (T - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T - \tau)^\alpha) d\tau - v_n^T. \end{aligned}$$

With the aim of reconstructing the initial value $\phi(r)$, we only need to solve the following integral equation

$$\int_0^R \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n T^\alpha) \zeta \phi(\zeta) \omega_n(\zeta) \omega_n(r) d\zeta = h(r). \quad (2.7)$$

By Lemma 2.2, in most cases, the equation (2.7) can be written as

$$\int_0^R \sum_{n=1, n \notin I_1}^{\infty} E_{\alpha,1}(-\lambda_n T^\alpha) \zeta \phi(\zeta) \omega_n(\zeta) \omega_n(r) d\zeta = h(r). \quad (2.8)$$

Similar to reference [17], the equation (2.8) in $L^2([0, R], r)$ has unique best approximate solution

$$\hat{\phi}(r) = \sum_{n=1, n \notin I_1}^{\infty} \frac{h_n}{|E_{\alpha,1}(-\lambda_n T^\alpha)|} \omega_n^*(r), \quad (2.9)$$

where

$$\omega_n^*(r) = \begin{cases} \omega_n(r), & E_{\alpha,1}(-\lambda_n T^\alpha) > 0, \\ -\omega_n(r), & E_{\alpha,1}(-\lambda_n T^\alpha) < 0. \end{cases}$$

From Lemma 2.3, we have

$$\frac{1}{|E_{\alpha,1}(-\lambda_n T^\alpha)|} \geq \frac{\lambda_n}{C} = \left(\frac{\xi_n}{R}\right)^2 \frac{1}{C} \rightarrow \infty, \quad n \rightarrow \infty,$$

which indicates that the inverse problem is ill-posed. Consequently, a regularization method is necessary to recover the stability of the solution of the inverse problem. Before that, we give a theorem.

Theorem 2.1: For any $\phi(r) \in H_p \cap \text{span}\{\omega_n(r), n \in I_1\}^\perp$, assume that there exists an a priori bound condition E such that

$$\|\phi(r)\|_{H_p} \leq E, \quad p > 0, \quad (2.10)$$

then we have

$$\|\phi(r)\| \leq C_1 E^{\frac{2}{p+2}} \|h(r)\|^{\frac{p}{p+2}}, \quad p > 0, \quad (2.11)$$

where $C_1 = (\frac{1}{\underline{C}})^{\frac{p}{p+2}}$ is a constant.

Proof: For any $\phi(r) \in \text{span}\{\omega_n(r), n \in I_1\}^\perp$, we have

$$\phi(r) = \sum_{n=1, n \notin I_1}^{\infty} \phi_n \omega_n(r).$$

Using the Hölder inequality, we obtain

$$\begin{aligned} \|\phi(r)\|^2 &= \sum_{n=1, n \notin I_1}^{\infty} \frac{h_n^2}{|E_{\alpha,1}(-\lambda_n T^\alpha)|^2} \\ &= \sum_{n=1, n \notin I_1}^{\infty} \frac{h_n^{\frac{4}{p+2}}}{|E_{\alpha,1}(-\lambda_n T^\alpha)|^2} h_n^{\frac{2p}{p+2}} \\ &\leq \left(\sum_{n=1, n \notin I_1}^{\infty} \frac{h_n^2}{|E_{\alpha,1}(-\lambda_n T^\alpha)|^{p+2}} \right)^{\frac{2}{p+2}} \left(\sum_{n=1, n \notin I_1}^{\infty} h_n^2 \right)^{\frac{p}{p+2}}. \end{aligned} \quad (2.12)$$

From Lemma 2.3, there holds

$$\begin{aligned} \sum_{n=1, n \notin I_1}^{\infty} \frac{h_n^2}{|E_{\alpha,1}(-\lambda_n T^\alpha)|^{p+2}} &\leq \sum_{n=1, n \notin I_1}^{\infty} \frac{h_n^2}{|E_{\alpha,1}(-\lambda_n T^\alpha)|^2} \left(\frac{\lambda_n}{\underline{C}}\right)^p \\ &= \sum_{n=1, n \notin I_1}^{\infty} \lambda_n^p \phi_n^2 \left(\frac{1}{\underline{C}}\right)^p = \left(\frac{1}{\underline{C}}\right)^p \|\phi(r)\|_{H_p}^2 \leq \left(\frac{1}{\underline{C}}\right)^p E^2. \end{aligned} \quad (2.13)$$

Combining (2.12) and (2.13), we can get that

$$\|\phi(r)\|^2 \leq \left(\frac{1}{\underline{C}}\right)^{\frac{2p}{p+2}} E^{\frac{4}{p+2}} \|h(r)\|^{\frac{2p}{p+2}}.$$

3. Iterative regularization method and error estimate

In this section, an iterative regularization method is proposed to solve the inverse problem. Besides, we give the error estimates under the a priori and a posteriori regularization parameter choice rules respectively.

We construct the following direct problem to approximate the solution of the inverse problem for $d = 2$, and $u^k(r, t)$ is the solution of this problem.

$$\begin{cases} D_t^\alpha u^k(r, t) = u_{rr}^k(r, t) + \frac{1}{r} u_r^k(r, t) + f(r, t), & 0 < r < R, 0 < t < T, \\ u^k(r, 0) = \phi^{k,\delta}(r), & 0 \leq r \leq R, \\ u_t^k(r, 0) = \psi(r), & 0 \leq r \leq R, \\ u_r^k(R, t) + \beta u^k(R, t) = \mu(t), & 0 \leq t \leq T, \end{cases}$$

where $\phi^{k,\delta}(r)$ has the iteration formula

$$\phi^{0,\delta}(r) = 0, \quad \phi^{k,\delta}(r) = \phi^{k-1,\delta}(r) - s\mathcal{A}^\gamma(u^{k-1}(r, T) - g^\delta(r)), \quad k = 1, 2, 3, \dots, \quad (3.1)$$

where k is the number of iterations which is equivalent to the regularization parameter, s is a real number satisfying $0 < s|E_{\alpha,1}(-\lambda_n T^\alpha)| < 1$ for all $n \in \mathbb{N}$, $0 \leq \gamma \leq 1$ is a real number, and \mathcal{A} is an operator satisfying $\mathcal{A}^\gamma h(r) = \sum_{n=1}^{\infty} E_{\alpha,1}^\gamma(-\lambda_n T^\alpha) h_n \omega_n(r)$. If $\gamma = 0$, the iterative formula (3.1) is the iterative regularization formula proposed by wang and Ran in reference [14], which means that our proposed iterative formula is more general.

Denote $\phi_n^{k,\delta} = (\phi^{k,\delta}(r), \omega_n(r))$, we can easily get

$$\begin{aligned} u^k(r, T) &= \sum_{n=1}^{\infty} [(\phi_n^{k,\delta} - v_n^0) E_{\alpha,1}(-\lambda_n T^\alpha) + \tilde{\psi}_n T E_{\alpha,2}(-\lambda_n T^\alpha) \\ &\quad + \int_0^T \tilde{f}_n(\tau) (T - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T - \tau)^\alpha) d\tau + v_n^T] \omega_n(r), \end{aligned}$$

According to the iteration formula (3.1), we have

$$\begin{aligned} \phi^{k,\delta}(r) &= \sum_{n=1}^{\infty} [\phi_n^{k-1,\delta} - s E_{\alpha,1}^\gamma(-\lambda_n T^\alpha) ((\phi_n^{k-1,\delta} - v_n^0) E_{\alpha,1}(-\lambda_n T^\alpha) + \tilde{\psi}_n T E_{\alpha,2}(-\lambda_n T^\alpha) \\ &\quad + \int_0^T \tilde{f}_n(\tau) (T - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T - \tau)^\alpha) d\tau + v_n^T - g_n^\delta)] \omega_n(r) \\ &= \sum_{n=1}^{\infty} [\phi_n^{k-1,\delta} - s E_{\alpha,1}^\gamma(-\lambda_n T^\alpha) (\phi_n^{k-1,\delta} E_{\alpha,1}(-\lambda_n T^\alpha) - h_n^\delta)] \omega_n(r) \\ &= \sum_{n=1}^{\infty} [(1 - s E_{\alpha,1}^{1+\gamma}(-\lambda_n T^\alpha)) \phi_n^{k-1,\delta} + s E_{\alpha,1}^\gamma(-\lambda_n T^\alpha) h_n^\delta] \omega_n(r) \\ &= \sum_{n=1}^{\infty} [(1 - s E_{\alpha,1}^{1+\gamma}(-\lambda_n T^\alpha))^k \phi_n^{0,\delta} + \sum_{i=0}^{k-1} (1 - s E_{\alpha,1}^{1+\gamma}(-\lambda_n T^\alpha))^i s E_{\alpha,1}^\gamma(-\lambda_n T^\alpha) h_n^\delta] \omega_n(r) \\ &= \sum_{n=1}^{\infty} \left(\sum_{i=0}^{k-1} (1 - s E_{\alpha,1}^{1+\gamma}(-\lambda_n T^\alpha))^i s E_{\alpha,1}^\gamma(-\lambda_n T^\alpha) h_n^\delta \right) \omega_n(r), \end{aligned} \quad (3.2)$$

where $g_n^\delta = (g(r)^\delta, \omega_n(r))_r$ and $h_n^\delta = g_n^\delta + v_n^0 E_{\alpha,1}(-\lambda_n T^\alpha) - v_n^T - \tilde{\psi}_n T E_{\alpha,2}(-\lambda_n T^\alpha) - \int_0^T \tilde{f}_n(\tau) (T - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T - \tau)^\alpha) d\tau$.

Noting that $(1 - s E_{\alpha,1}^{1+\gamma}(-\lambda_n T^\alpha))^i$ in (3.2) may tend to infinity or be meaningless when $E_{\alpha,1}(-\lambda_n T^\alpha) < 0$, so we modify the regularization solution approximation solution (3.2) as

$$\hat{\phi}^{k,\delta}(r) = \sum_{n=1, n \notin I_1}^{\infty} \left(\sum_{i=0}^{k-1} (1 - s|E_{\alpha,1}(-\lambda_n T^\alpha)|^{1+\gamma})^i s|E_{\alpha,1}(-\lambda_n T^\alpha)|^\gamma h_n^\delta \right) \omega_n^*(r). \quad (3.3)$$

From (3.3), we can find that when $\gamma = 1$, (3.3) can be transformed into the classical Landweber iterative method [15], and when $0 < \gamma < 1$, (3.3) can be transformed into the

modified fractional Landweber iterative method [9]. Both Landweber iterative methods have well-established theories, and we can perform theoretical analysis similarly. For the sake of simplicity, we analyze for the case of $\gamma = 0$.

For $\gamma = 0$, the regularized approximation solution is

$$\widehat{\phi}^{k,\delta}(r) = \sum_{n=1, n \notin I_1}^{\infty} \left(\sum_{i=0}^{k-1} (1 - \sigma_n)^i s h_n^\delta \right) \omega_n^*(r), \quad (3.4)$$

where $\sigma_n = s|E_{\alpha,1}(-\lambda_n T^\alpha)|$.

3.1. a priori regularization parameter choice rule

Theorem 3.1: Let $\widehat{\phi}(r)$ be given by (2.9) and $\widehat{\phi}^{k,\delta}(r)$ be given by (3.4). Assume that $\widehat{\phi}(r)$ satisfies a priori bound condition (2.10) and the assumption (1.9) holds, then we have the regularization parameter

$$k = \left[\frac{1}{s} \left(\frac{E}{\delta} \right)^{\frac{2}{p+2}} \right], \quad (3.5)$$

and the error estimate

$$\|\widehat{\phi}^{k,\delta}(r) - \widehat{\phi}(r)\| \leq (1 + \theta_{\frac{p}{2}} \underline{C}^{-\frac{p}{2}}) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, \quad (3.6)$$

where $[x]$ represents a maximum integer not exceeding x and \underline{C} is a constant.

Proof: By the triangle inequality, we have

$$\|\widehat{\phi}^{k,\delta}(r) - \widehat{\phi}(r)\| \leq \|\widehat{\phi}^{k,\delta}(r) - \widehat{\phi}^k(r)\| + \|\widehat{\phi}^k(r) - \widehat{\phi}(r)\|.$$

By (1.9) and (3.4), and then applying Lemma 2.4 (taking $\mu = 0$), we can obtain that

$$\begin{aligned} \|\widehat{\phi}^{k,\delta}(r) - \widehat{\phi}^k(r)\| &= \left\| \sum_{n=1, n \notin I_1}^{\infty} p_k(\sigma_n) s (g_n^\delta - g_n) \omega_n^*(r) \right\| \\ &\leq \sup_{n \in \mathbb{N}} (p_k(\sigma_n) s) \|g^\delta(r) - g(r)\| \leq \delta \sup_{n \in \mathbb{N}} (p_k(\sigma_n) s) \leq s k \delta. \end{aligned} \quad (3.7)$$

Using the a priori bound condition (2.10), Lemma 2.3 and Lemma 2.4 (letting $\nu = \frac{p}{2}$), there holds

$$\begin{aligned} \|\widehat{\phi}^k(r) - \widehat{\phi}(r)\| &= \left\| \sum_{n=1, n \notin I_1}^{\infty} (-r_k(\sigma_n)) \frac{h_n}{|E_{\alpha,1}(-\lambda_n T^\alpha)|} \omega_n^*(r) \right\| \\ &= \left\| \sum_{n=1, n \notin I_1}^{\infty} r_k(\sigma_n) \widehat{\phi}_{nn} \omega_n^*(r) \right\| \\ &= \left(\sum_{n=1, n \notin I_1}^{\infty} r_k^2(\sigma_n) (\lambda_n^{-p}) (\lambda_n^p) \widehat{\phi}_{nn}^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{n \in \mathbb{N}} (r_k(\sigma_n)(\lambda_n^{-\frac{p}{2}})) \left(\sum_{n=1, n \notin I_1}^{\infty} (\lambda_n^p) \widehat{\phi}_n^2 \right)^{\frac{1}{2}} \\
&\leq E \sup_{n \in \mathbb{N}} (r_k(\sigma_n)(\lambda_n^{-\frac{p}{2}})) \\
&\leq E \sup_{n \in \mathbb{N}} (r_k(\sigma_n) \underline{C}^{-\frac{p}{2}} |E_{\alpha,1}(-\lambda_n T^\alpha)|^{\frac{p}{2}}) \\
&= E (\underline{sC})^{-\frac{p}{2}} \sup_{n \in \mathbb{N}} (r_k(\sigma_n) \sigma_n^{\frac{p}{2}}) \\
&\leq \theta_{\frac{p}{2}} (\underline{sC})^{-\frac{p}{2}} E(k+1)^{-\frac{p}{2}}.
\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), choosing $k = \lceil \frac{1}{s} (\frac{E}{\delta})^{\frac{2}{p+2}} \rceil$, we obtain (3.6).

3.2. a posteriori regularization parameter choice rule

Next, the rule of a posteriori regularization parameter choice is given by the discrepancy principle.

Define the an orthogonal operator $P : L^2([0, R], r) \rightarrow \text{span}\{\omega_n(r), n \in I_1\}^\perp$, we have

$$\|Ph^\delta(r) - Ph(r)\| \leq \|h^\delta(r) - h(r)\| = \|g^\delta(r) - g(r)\| \leq \delta.$$

Then the regularization parameter to satisfy

$$\left\| \sum_{n=1, n \notin I_1}^{\infty} p_k(\sigma_n) \sigma_n h_n^\delta \omega_n^*(r) - Ph^\delta(r) \right\| \leq \tau \delta < \left\| \sum_{n=1, n \notin I_1}^{\infty} p_{k-1}(\sigma_n) \sigma_n h_n^\delta \omega_n^*(r) - Ph^\delta(r) \right\|, \tag{3.9}$$

where $\tau > 1$ is a constant.

Theorem 3.2: Let $\widehat{\phi}(r)$ be given by (2.9) and $\widehat{\phi}^{k,\delta}(r)$ be given by (3.4). Assume that $\widehat{\phi}(r, z)$ satisfies a priori bound condition (2.10) and the assumption (1.9) holds. The regularization parameter k is chosen by (3.9), then we have the following error estimate

$$\|\widehat{\phi}^{k,\delta}(r) - \widehat{\phi}(r)\| \leq (C_1(1+\tau)^{\frac{p}{p+2}} + (\frac{C^{-\frac{p}{2}} \theta_{\frac{p+2}{2}}}{\tau - 1})^{\frac{2}{p+2}}) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{3.10}$$

Proof: Using the triangle inequality, we have

$$\|\widehat{\phi}^{k,\delta}(r) - \widehat{\phi}(r)\| \leq \|\widehat{\phi}^{k,\delta}(r) - \widehat{\phi}^k(r)\| + \|\widehat{\phi}^k(r) - \widehat{\phi}(r)\|.$$

Similar to (3.7), it is easy to get that

$$\|\widehat{\phi}^{k,\delta}(r) - \widehat{\phi}^k(r)\| \leq sk\delta. \tag{3.11}$$

By (3.9) and (1.9), then applying Lemma 2.3 and Lemma 2.4 (taking $\nu = \frac{p+2}{2}$), there holds

$$\begin{aligned}
\tau\delta &< \left\| \sum_{n=1, n \notin I_1}^{\infty} p_{k-1}(\sigma_n) \sigma_n h_n^{\delta} \omega_n^*(r) - Ph^{\delta}(r) \right\| \\
&= \left\| \sum_{n=1, n \notin I_1}^{\infty} r_{k-1}(\sigma_n) h_n^{\delta} \omega_n^*(r) \right\| \\
&\leq \left\| \sum_{n=1, n \notin I_1}^{\infty} r_{k-1}(\sigma_n) (h_n^{\delta} - h_n) \omega_n^*(r) \right\| + \left\| \sum_{n=1, n \notin I_1}^{\infty} r_{k-1}(\sigma_n) h_n \omega_n^*(r) \right\| \\
&\leq \delta + \left\| \sum_{n=1, n \notin I_1}^{\infty} r_{k-1}(\sigma_n) \widehat{\phi}_n |E_{\alpha,1}(-\lambda_n T^{\alpha})| \omega_n^*(r) \right\| \\
&= \delta + \left\| \sum_{n=1, n \notin I_1}^{\infty} r_{k-1}(\sigma_n) |E_{\alpha,1}(-\lambda_n T^{\alpha})| (\lambda_n)^{-\frac{p}{2}} (\lambda_n)^{\frac{p}{2}} \widehat{\phi}_n \omega_n^*(r) \right\| \\
&\leq \delta + E \sup_{n \in \mathbb{N}} (r_{k-1}(\sigma_n) |E_{\alpha,1}(-\lambda_n T^{\alpha})| (\lambda_n)^{-\frac{p}{2}}) \\
&\leq \delta + \underline{C}^{-\frac{p}{2}} \theta_{\frac{p+2}{2}} E(sk)^{-\frac{p+2}{2}}.
\end{aligned} \tag{3.12}$$

By (3.12), we deduce that

$$sk \leq \left(\frac{\underline{C}^{-\frac{p}{2}} \theta_{\frac{p+2}{2}}}{\tau - 1} \right)^{\frac{2}{p+2}} \left(\frac{E}{\delta} \right)^{\frac{2}{p+2}}. \tag{3.13}$$

Substituting (3.13) into (3.11), we have

$$\|\widehat{\phi}^{k,\delta}(r) - \widehat{\phi}^k(r)\| \leq \left(\frac{\underline{C}^{-\frac{p}{2}} \theta_{\frac{p+2}{2}}}{\tau - 1} \right)^{\frac{2}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{3.14}$$

Furthermore, depending on $\widehat{\phi}_n^k = (\widehat{\phi}^k(r), \omega_n^*(r))$ and $\widehat{\phi}_n = (\widehat{\phi}(r), \omega_n^*(r))$, we have

$$\begin{aligned}
&\left\| \sum_{n=1, n \notin I_1}^{\infty} |E_{\alpha,1}(-\lambda_n T^{\alpha})| (\widehat{\phi}_n^k - \widehat{\phi}_n) \omega_n^*(r) \right\| \\
&= \left\| \sum_{n=1, n \notin I_1}^{\infty} r_k(\sigma_n) h_n \omega_n^*(r) \right\| \\
&\leq \left\| \sum_{n=1, n \notin I_1}^{\infty} r_k(\sigma_n) (h_n - h_n^{\delta}) \omega_n^*(r) \right\| + \left\| \sum_{n=1, n \notin I_1}^{\infty} r_k(\sigma_n) h_n^{\delta} \omega_n^*(r) \right\| \\
&\leq \delta + \tau\delta = (1 + \tau)\delta.
\end{aligned} \tag{3.15}$$

From (2.10), there holds

$$\|\widehat{\phi}^k(r) - \widehat{\phi}(r)\|_{H_p} = \left\| \sum_{n=1, n \notin I_1}^{\infty} (\lambda_n)^{\frac{p}{2}} r_k(\sigma_n) \frac{h_n}{|E_{\alpha,1}(-\lambda_n T^{\alpha})|} \omega_n^*(r) \right\|$$

$$\leq \left\| \sum_{n=1, n \notin I_1}^{\infty} (\lambda_n)^{\frac{p}{2}} \widehat{\phi}_n \omega_n^*(r) \right\| \leq E.$$

By Theorem 2.1 and (3.15), we deduce that

$$\|\widehat{\phi}^k(r) - \widehat{\phi}(r)\| \leq C_1(1 + \tau)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \quad (3.16)$$

Combining (3.14) and (3.16), we obtain (3.10).

4. Numerical implementations

Now we give some examples for $d = 2$ to show the effectiveness and stability of the proposed regularization methods although we focus on the numerical theoretical analysis. To avoid the 'inverse crime', the finite difference methods [20] are used to calculate the forward problem. The finite difference schemes are sketched as follows.

Firstly, we denote the grid points in the time interval $[0, T]$ as $t_n = n\tau$ ($n = 0, 1, \dots, N$) with the time step size $\tau = \frac{T}{N}$, the grid points in the space interval $[0, R]$ as $r_i = ih$ ($i = 0, 1, \dots, M$) with the space step size $h = \frac{R}{M}$. The approximate values of each grid point is $u_i^n = u(r_i, t_n)$.

Secondly, based on the finite difference scheme, the equation $D_t^\alpha u(r, t) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + f(r, t)$ can be written as

$$\begin{aligned} & \left(\frac{q}{4r_i} - \frac{p}{2}\right)u_{i-1}^n + (1+p)u_i^n - \left(\frac{q}{4r_i} + \frac{p}{2}\right)u_{i+1}^n \\ &= \left(\frac{p}{2} - \frac{q}{4r_i}\right)u_{i-1}^{n-1} + (1-p)u_i^{n-1} + \left(\frac{q}{4r_i} + \frac{p}{2}\right)u_{i+1}^{n-1} \\ &+ \sum_{k=1}^{n-1} (b_{n-k-1}^{(\alpha)} - b_{n-k}^{(\alpha)})(u_i^k - u_i^{k-1}) + \tau b_{n-1}^{(\alpha)} \psi_i + \frac{\tau^\alpha \Gamma(3-\alpha)}{2} (f_i^n + f_i^{n-1}), \end{aligned}$$

where $b_k^{(\alpha)} = (k+1)^{2-\alpha} - k^{2-\alpha}$ ($k \geq 0$), $p = \frac{\tau^\alpha \Gamma(3-\alpha)}{h^2}$, $q = \frac{\tau^\alpha \Gamma(3-\alpha)}{h}$, and the Robin boundary condition $\frac{\partial u}{\partial r} + \beta u = \mu(t)$ can be written as

$$\begin{aligned} & -pu_{M-1}^n + (1+p + \frac{qh\beta}{2r_M} + ph\beta)u_M^n \\ &= pu_{M-1}^{n-1} + (1-p - \frac{qh\beta}{2r_M} - ph\beta)u_M^{n-1} + \sum_{k=1}^{n-1} (b_{n-k-1}^{(\alpha)} - b_{n-k}^{(\alpha)})(u_M^k - u_M^{k-1}) \\ &+ \tau b_{n-1}^{(\alpha)} \psi_M + \frac{\tau^\alpha \Gamma(3-\alpha)}{2} (f_M^n + f_M^{n-1}) + h(\frac{q}{2r_M} + \frac{p}{2})(\mu(t_n) + \mu(t_{n-1})). \end{aligned}$$

Finally, the final data $u(r, T) = g(r)$ can be obtained.

After obtaining the final data $g(r)$, we generate the noisy data $g^\delta(r)$ by

$$g^\delta = g + \epsilon g \cdot \text{randn}(\text{size}(g)),$$

the function $randn(\cdot)$ generates arrays of random numbers whose elements are normally distributed with zero mean and unit standard deviation. The corresponding noise level is calculated by $\delta = \|g^\delta - g\|$.

To illustrate the accuracy of the regularized solution, we calculate the relative error between the exact solution and the regularized solution by

$$e_r(\hat{\phi}, \epsilon) = \frac{\|\hat{\phi}^{k,\delta}(r) - \hat{\phi}(r)\|}{\|\hat{\phi}(r)\|}.$$

In the numerical experiments, we always fix $R = 2$, $T = 1$, $\beta = 1$, $N = 20$, $M = 50$. The regularization parameter k under the a priori choice rule is given by (3.4), and the regularization parameter k under the a posteriori choice rule is chosen by (3.8) with $\tau = 1.1$.

Example 4.1: Take the initial values

$$\begin{aligned}\phi(r) &= (r^2 - 4), \\ \psi(r) &= 0,\end{aligned}$$

the source term

$$f(r, t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}(r^2 - 4) - 4(t^2 + 1),$$

and the Robin boundary value

$$\mu(t) = 4(t^2 + 1).$$

Then the exact solution is given by

$$u(r, t) = (t^2 + 1)(r^2 - 4).$$

Example 4.2: Take the initial values

$$\begin{aligned}\phi(r) &= \begin{cases} \frac{1}{2} + \frac{r}{2}, & 0 \leq r < 1, \\ 2 - r, & 1 \leq r \leq 2, \end{cases} \\ \psi(r) &= 0,\end{aligned}$$

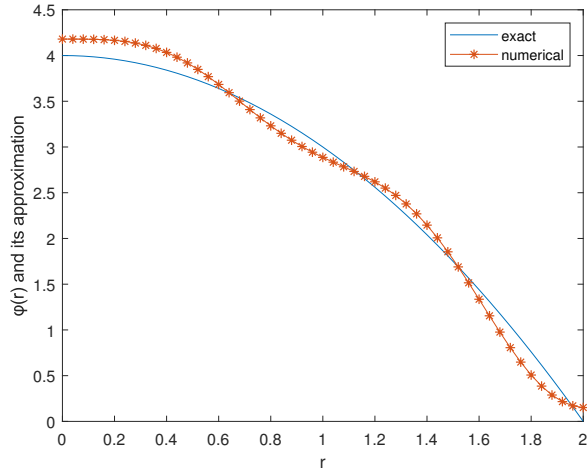
the source term

$$f(r, t) = 2t^{2-\alpha}r,$$

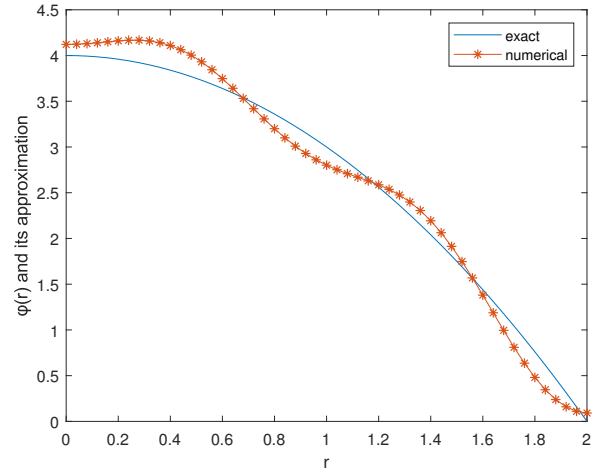
and the Robin boundary value

$$\mu(t) = -(t^2 + 1).$$

Figures 2-5 give the comparisons between the exact solution $\phi(r)$ and its regularized approximate solution $\widehat{\phi}^{k,\delta}(r)$ under the a priori and a posteriori parameter choice rules with noise level $\epsilon = 0.001$ for Example 4.1-4.2. Tables 1-2 show the relative errors with different noise levels of Example 4.1-4.2 with $\alpha = 1.5$.

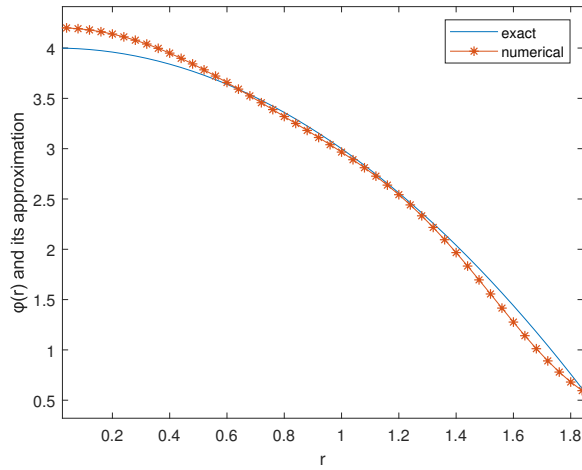


(a) *A priori parameter choice rule*

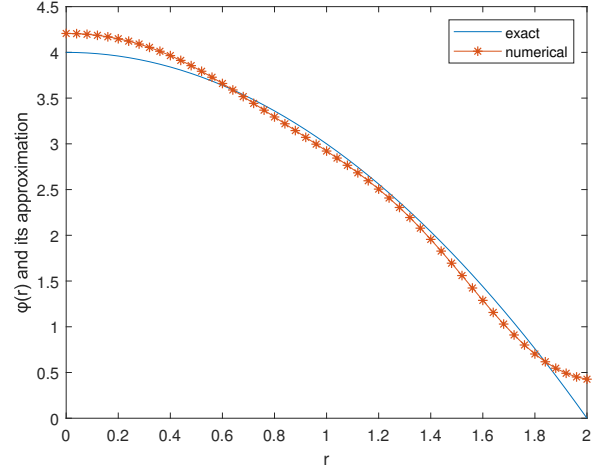


(b) *A posteriori parameter choice rule*

Figure 2: The exact solution and the regularized approximate solution for Example 4.1 with $\alpha = 1.5$.



(a) *A priori parameter choice rule*

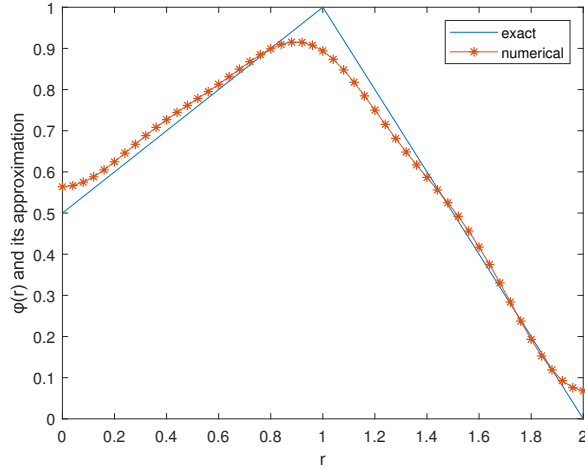


(b) *A posteriori parameter choice rule*

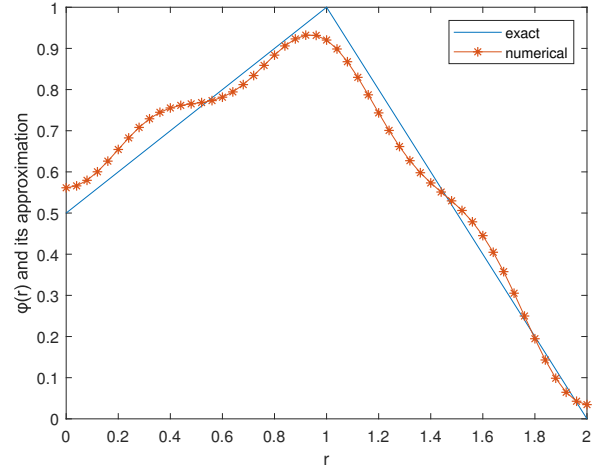
Figure 3: The exact solution and the regularized approximate solution for Example 4.1 with $\alpha = 1.8$.

Table 1: Numerical results of Example 4.1 for different ϵ with $\alpha = 1.5$.

ϵ	0.001	0.005	0.01
$e_r(\hat{\phi}, \epsilon)_{priori}$	0.0504	0.0797	0.0846
$e_r(\hat{\phi}, \epsilon)_{posteriori}$	0.0591	0.0746	0.0933

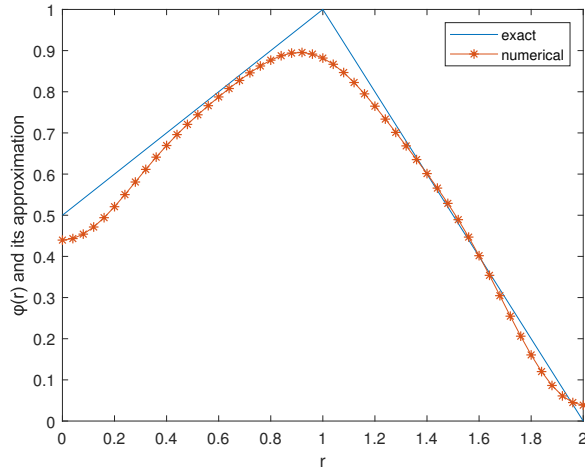


(a) *A priori parameter choice rule*

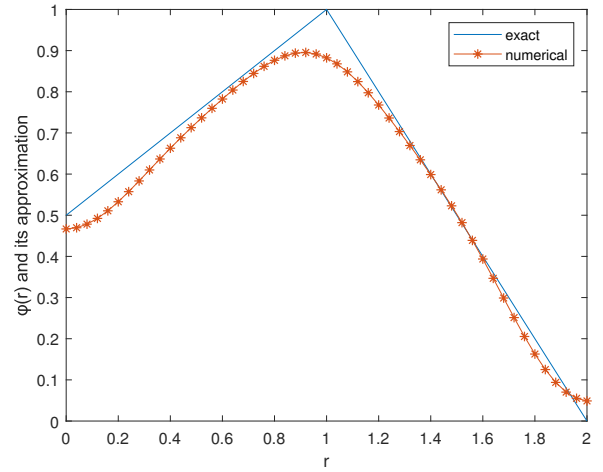


(b) *A posteriori parameter choice rule*

Figure 4: The exact solution and the regularized approximate solution for Example 4.2 with $\alpha = 1.5$.



(a) *A priori parameter choice rule*



(b) *A posteriori parameter choice rule*

Figure 5: The exact solution and the regularized approximate solution for Example 4.2 with $\alpha = 1.8$.

Table 2: Numerical results of Example 4.2 for different ϵ with $\alpha = 1.5$

ϵ	0.001	0.005	0.01
$e_r(\hat{\phi}, \epsilon)_{priori}$	0.0547	0.1443	0.2609
$e_r(\hat{\phi}, \epsilon)_{posteriori}$	0.0619	0.1620	0.2852

From the two Examples, it can be seen that the smaller ϵ , the better the numerical results. Besides, we can conclude that the a posteriori parameter choice rule is even comparable to the a priori parameter choice rule. The numerical examples show that the proposed iterative regularization method is effective and stable.

5. Conclusion

In this paper, we consider the backward problem for radially symmetric time-fractional diffusion-wave equation under Robin boundary condition. We present an iterative regularization method to obtain the regularized approximate solution, and error estimates are given under two parameter choice rules. The numerical examples are conducted for showing the effectiveness and stability of the proposed method.

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