Sandia National Laboratories

Eigenvalue Uncoupling of Electrokinetic Flows

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There is a need to solve coupled physics problems efficiently. We present a simple approach to uncoupling electrokinetic flow using a matrix technique (Lo et al., 2009). Electrokinetics is movement of ions and water in a porous medium under fluid pressure (p) and electrostatic potential (ψ) gradients:

$$\begin{bmatrix} C^* & 0 \\ 0 & nc \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \psi \\ p \end{bmatrix} = \begin{bmatrix} \sigma_0 & L_{12} \\ L_{21} & \frac{k_0}{\mu} \end{bmatrix} \nabla^2 \begin{bmatrix} \psi \\ p \end{bmatrix}$$

Consolidating coefficients leads to:

$$\frac{\partial}{\partial t} \begin{bmatrix} \psi \\ p \end{bmatrix} = \begin{bmatrix} \alpha_E & \alpha_E K_S \\ \alpha_H K_E & \alpha_H \end{bmatrix} \nabla^2 \begin{bmatrix} \psi \\ p \end{bmatrix}$$

Non-dimensionalizing the equations reduces free parameters from 4 to 2, the diffusivity ratio ($\alpha_D = \alpha_E/\alpha_H$) and the electrokinetic coupling strength ($K_D = K_E K_S$):

$$\frac{\partial}{\partial t_D} \begin{bmatrix} \psi_D \\ p_D \end{bmatrix} = \overset{\mathbf{A}}{\begin{bmatrix} \alpha_D & \alpha_D \\ K_D & 1 \end{bmatrix}} \nabla_D^2 \underset{\mathbf{d}_D}{\begin{bmatrix} \psi_D \\ p_D \end{bmatrix}}$$

Using the eigenvalue decomposition from linear algebra:

$$\left.\begin{aligned} \frac{\partial \mathbf{d}_D}{\partial t} &= \mathbf{A}\nabla_D^2 \mathbf{d}_D \\ \mathbf{A} &= \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1} \end{aligned}\right\} \frac{\partial}{\partial t_D}\underbrace{\mathbf{S}^{-1}\mathbf{d}_D}_{\boldsymbol{\delta}} = \overbrace{(\mathbf{S}^{-1}\mathbf{S})}^{\boldsymbol{I}}\boldsymbol{\Lambda}\nabla_D^2\underbrace{\mathbf{S}^{-1}\mathbf{d}_D}_{\boldsymbol{\delta}}$$

We get two uncoupled governing equations (diagonal $\boldsymbol{\Lambda}$ matrix), but in terms of a vector of new potentials ($\boldsymbol{\delta} = \mathbf{S}^{-1}\mathbf{d}_D$). Eigenvector matrices ($\boldsymbol{S}$) and eigenvalue matrix ($\boldsymbol{\Lambda}$) are below.

<u>Uncoupling Procedure:</u>

1) Pose problem in terms of δ_1, δ_2
2) Solve 2 uncoupled diffusion problems
3) Compute p, ψ via matrix multiply ($\mathbf{d}_D = \mathbf{S}\boldsymbol{\delta}$)

$$\mathbf{S} = \begin{bmatrix} \frac{2\alpha_D}{1-\alpha_D-\Delta} & -\frac{1}{2K_D}(1-\alpha_D-\Delta) \\ 1 & 1 \end{bmatrix}$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \frac{2\alpha_D(1-K_D)}{1+\alpha_D+\Delta} & 0 \\ 0 & \frac{1}{2}(1+\alpha_D+\Delta) \end{bmatrix}$$

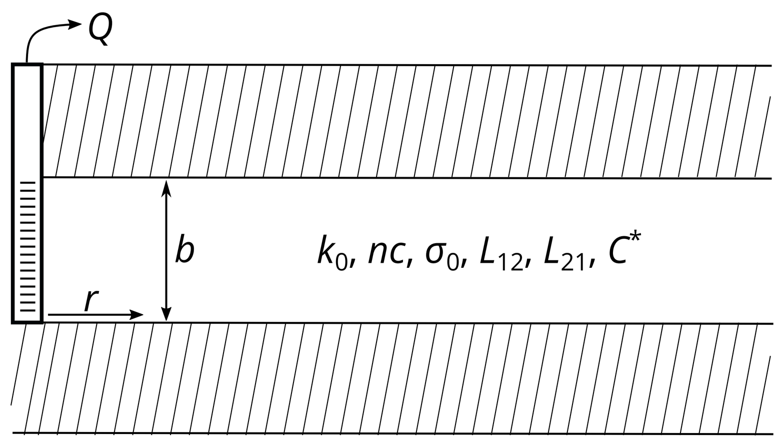
$$\mathbf{S}^{-1} = \frac{1}{\Delta}\begin{bmatrix} -K_D & -\frac{1}{2}(1-\alpha_D-\Delta) \\ K_D & \frac{-2K_D\alpha_D}{1-\alpha_D-\Delta} \end{bmatrix}$$

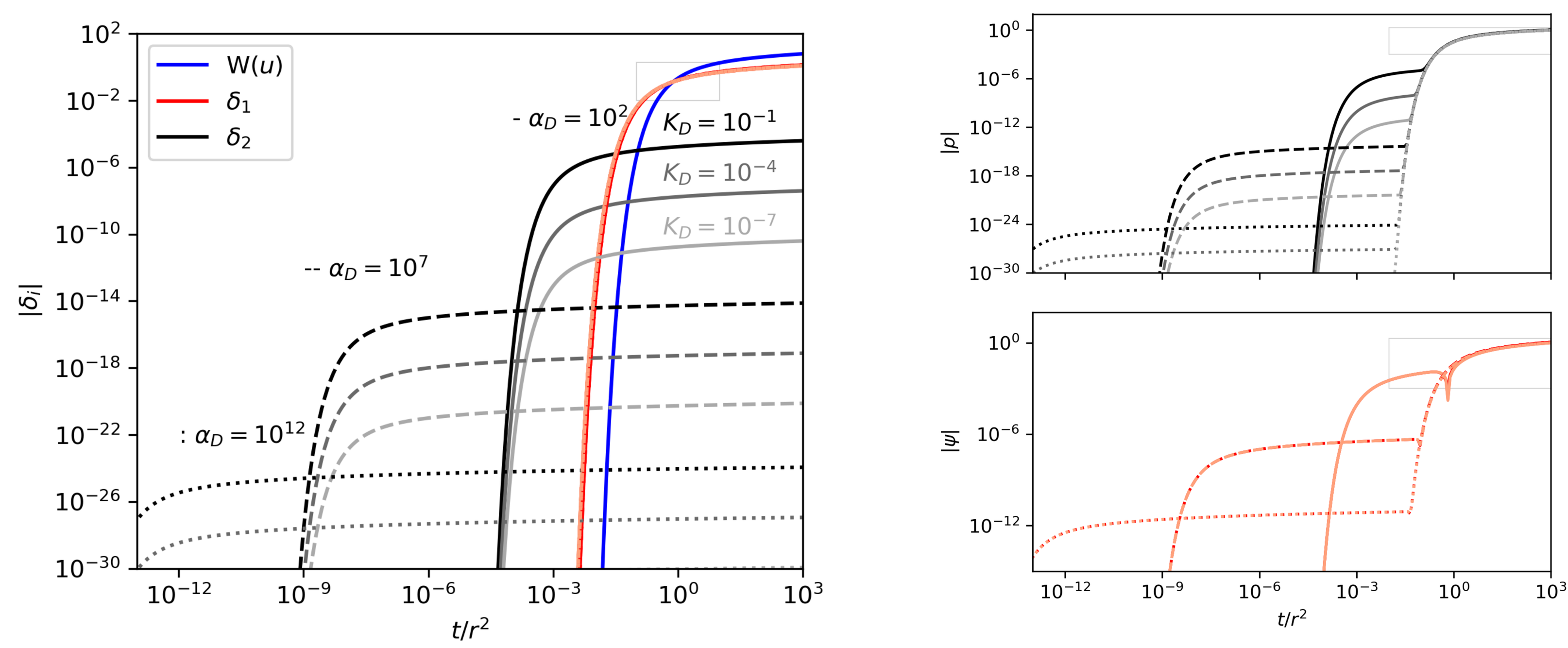
$$\Delta = \sqrt{1+\alpha_D(4K_D-2+\alpha_D)}$$

Lo, Sposito & Majer, 2009. Analytical decoupling of poroelasticity equations for acoustic-wave propagation and attenuation in a porous medium containing two immiscible fluids. *Journal of Engineering Mathematics*, 64:219–235.

Example 1. Flow to Pumping Well (Theis)

1D radial flow to a confined well surrounded by insulators for electrokinetic problem. The solution to the decoupled intermediate equations (δ_1, δ_2) comes from the Theis (1935) well function (exponential integral).





Initial type curve (blue), δ_1 solutions (red), and δ_2 solutions (black) for a range of α_D and K_D (left). The solution for (p, ψ) is recombined (right) using $\mathbf{d}_D = \mathbf{S}\boldsymbol{\delta}$.

Using only *type curve* methods, we solve coupled streaming potential and electroosmosis.

x-axis: $$\overbrace{\ln\left[\frac{t}{r^2}\right]}^{\text{data}} = \overbrace{\ln\left[\frac{1}{4\Lambda_{ii}}\right]}^{\text{shift}} + \overbrace{\ln\left[\frac{1}{u}\right]}^{\text{type-curve}}$$

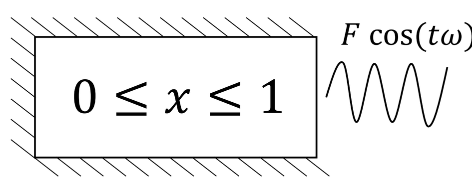
y-axis: $$\ln[\delta_i] = \ln\left[\frac{Q_i}{4\pi\Lambda_{ii}}\right] + \ln[\mathrm{E}_1(u)]$$

Variable	Description	Dimensions
α_H, α_E	hydraulic/electrical diffusivity	$\mathrm{m^2/sec}$
t	time	sec
k_0	formation permeability	$\mathrm{m^2}$
c	formation compressibility	1/Pa
n	formation porosity	–
C^*	electric capacitance of formation	$\mathrm{C/(m^3 \cdot V)}$
σ_0	electric conductivity	S/m
p	change in fluid pressure	Pa
ψ	change in electrostatic potential	V
μ	viscosity of water	$\mathrm{Pa \cdot sec}$
L_{12}, L_{21}	electrokinetic coupling coefficient	$\mathrm{A/(Pa \cdot m) = m^2/(V \cdot sec)}$
K_S	streaming potential	V/Pa
K_E	electroosmosis pressure	Pa/V

Theis, 1935. The relation between lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage. *Transactions, American Geophysical Union*, 16(2):519–524.

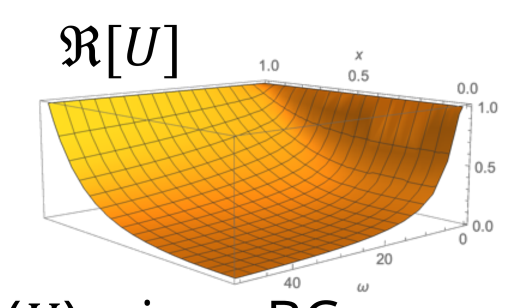
Example 2. Driven Electrokinetic Core

1D flow in a box driven by periodic Dirichlet BC on one end, no-flow / insulating on other end (Pengra et al., 1999). Steady-state solution is:



$$\delta_i(x_D,t_D) = \delta_{i\mathrm{SS}}(x_D,t_D) + \overset{0}{\cancel{\delta_{i\mathrm{TR}}(x_D,t_D)}},$$

$$\delta_{i\mathrm{SS}}(x_D,t_D) = \Re\left[U(x_D)e^{j\omega_D t_D}\right]$$



Solve ODE for complex amplitude (U), given BC

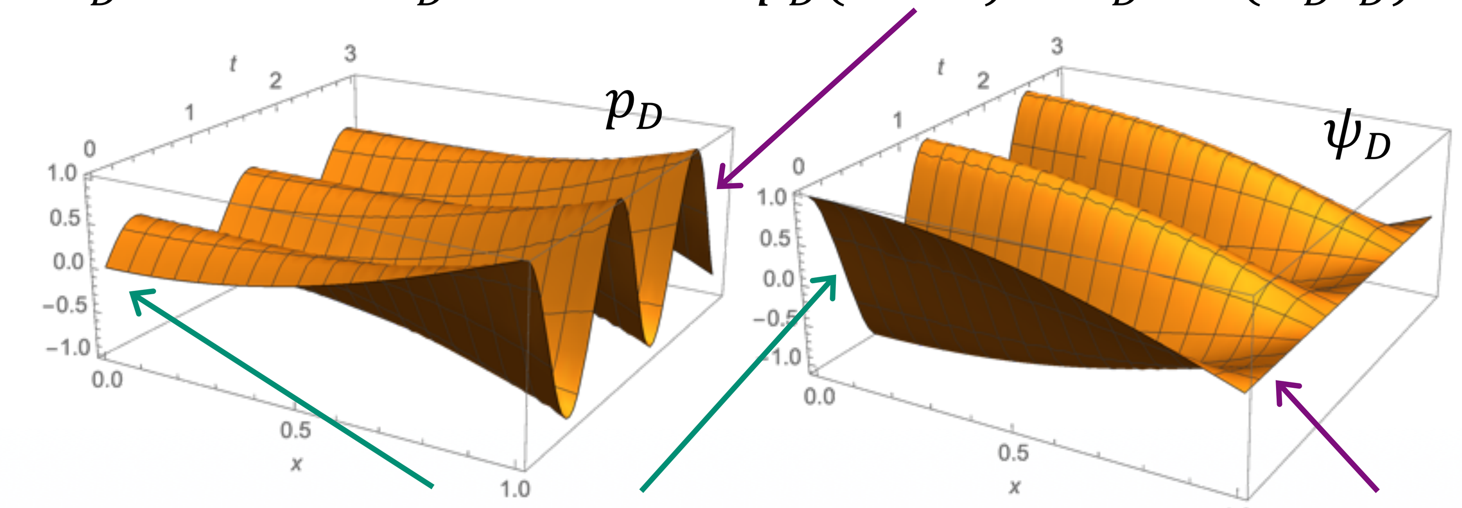
$$\frac{\mathrm{d}^2U}{\mathrm{d}x_D^2} - \frac{j\omega_D}{\Lambda_{ii}}U = 0 \qquad U(x_D) = c_1e^{\zeta x_D} + c_2e^{-\zeta x_D} \qquad \zeta = \sqrt{j\omega_D/\Lambda_{ii}}$$

$$\delta_{i\mathrm{SS}}(x_D,t_D) = \Re\left\{\frac{jK_DF_D\zeta}{\omega_D\Delta(e^{\zeta}-1)}\left[e^{i\omega_Dt_D+\zeta(1+x_D)} + e^{i\omega_Dt_D-\zeta x_D}\right]\right\}$$

Solution is applied to electrokinetics via decoupling

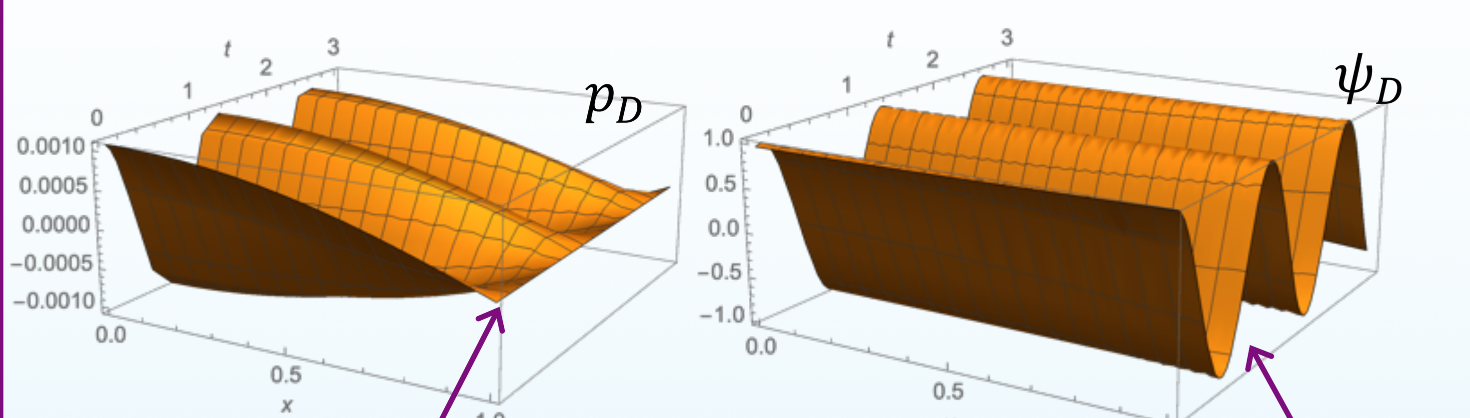
<u>Applied Pressure BC (Streaming Potential)</u>

$\alpha_D = 10$ $\quad K_D = 0.01$ $\quad p_D(x=1) = F_D\cos(\omega_D t_D)$



No-gradient BC ($x = 0$) $\quad \psi_D(x=1) = 0$

<u>Applied Voltage BC (Electroosmosis)</u>



$\psi_D(x=1) = 0$ $\quad \psi_D(x=1) = F_D\cos(\omega_D t_D)$

Pengra, Li & Wong, 1999. Determination of rock properties by low-frequency AC electrokinetics. *Journal of Geophysical Research*, 104(B12):29485-29508.

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