New upper bounds on the Gaussian Q-function via Jensen's inequality and integration by parts, and applications in symbol error probability analysis

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Using Jensen's inequality and integration by parts, we derive some tight upper bounds on the Gaussian Q-function. The tightness of the bounds obtained by Jensen's inequality can be improved by increasing the number of exponential terms, and one of them is invertible. We obtain a piece-wise upper bound and show its application in the analysis of the symbol error probability of various modulation schemes in different channel models.

Introduction: The Gaussian Q-function is of great significance in the field of communication. It is defined as

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt. \tag{1}$$

Its infinite integral form requires signal receivers to have higher processing and computing power and larger storage space. Consequently, many upper and lower bounds have been proposed in the literature to facilitate easy calculation.

The upper bound of Gaussian Q-function is obtained by integration by parts in [1]. Another form of Q-function was proposed in [2], which stimulated the research on further simplification and approximation of Q-function. According to the monotonically increasing property of the integrand in Craig form, the upper bound of the integrand is set as the maximum value in the range of integration, and a set of exponential upper bound of Q-function is obtained in [3]. A single-term, purely exponential, lower bound on the Gaussian Q-function is derived in [4], along with a method to optimize its tightness and a fast iterative method for its inversion.

The family of exponential lower bounds on the Gaussian Q-function [5] uses Jensen's inequality. We here propose to apply the same approach to the cumulative distribution function (CDF) of the standard Gaussian distribution in order to obtain a set of exponential upper bounds on the Gaussian Q-function, one of which is invertible. Our bounds have the similar simple form as the Chiani bounds in [3], but are tighter at low SNR values. A lower bound on the Gaussian Q-function [6, eq.(13)] was obtained via integration by parts. We here apply integration by parts again on the lower bound [6, eq.(13)] and get a new upper bound. Combining the two upper bounds, we obtain a tighter upper bound as a piecewise function. The accuracy of the new upper bound is proved by comparing it with the existing upper bounds of Chiani's[3] and Fu's[7].

As for applications, a closed-form upper bound of the inverse function is derived and some SEP expressions of various digital modulation techniques in different channel models are computed to justify the accuracy of our new upper bounds.

A piece-wise upper bound: The CDF of standard normal distribution is defined as

$$F(x) = 1 - Q(x) = \frac{1}{2} + \int_0^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt. \tag{2}$$

First, we split the integration range of $[0, x]$ into n subranges, by arbitrarily choosing $n+1$ values of β_k such that $0 = \beta_0 x < \beta_1 x < \ldots < \beta_k < \ldots < \beta_n x = x$, i.e. $\beta_0 = 0$ and $\beta_n = 1$. Thus, (2) becomes

$$F(x) = \frac{1}{2} + \sum_{k=1}^{n} \int_{\beta_{k-1}x}^{\beta_k x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt. \tag{3}$$

Then, we can apply the Jensen's inequality for each summation term in (3).

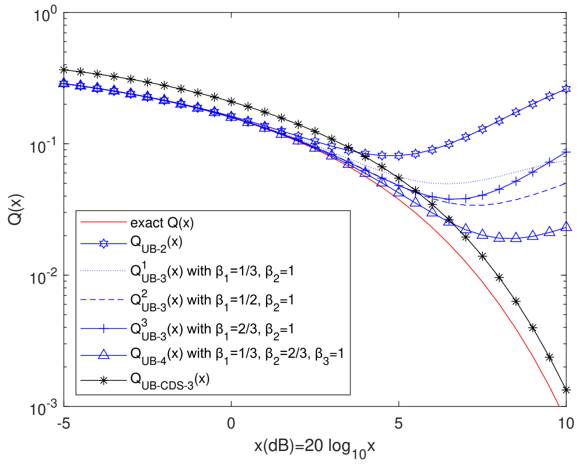

Fig 1 *Upper bounds on the Gaussian Q-function for small argument values.*

Jensen's inequality [5, eq.(3)]: Let $f(\theta)$ and $p(\theta)$ be two functions defined for $a \le \theta \le b$ such that $\alpha \le f(\theta) \le \beta$ and $p(\theta) \ge 0$, with $p(\theta) \not\equiv 0$. Let $\phi(u)$ be a convenx function defined on the interval $\alpha \le u \le \beta$, then

$$\phi\left(\frac{\int_a^b f(\theta)p(\theta)d\theta}{\int_a^b p(\theta)d\theta}\right) \le \frac{\int_a^b \phi(f)p(\theta)d\theta}{\int_a^b p(\theta)d\theta}. \tag{4}$$

Letting

$$\begin{aligned} \phi(u) &= \exp(u), \\ p(\theta) &= \frac{1}{\sqrt{2\pi}}, \\ f(\theta) &= -\frac{t^2}{2}, \\ a &= \beta_{k-1}x, \\ b &= \beta_k x, \end{aligned} \tag{5}$$

we obtain a lower bound on (3) as

$$F_{\mathrm{LB}}(x) = \frac{1}{2} + \sum_{k=1}^{n} a_k x \exp\left(-b_k x^2\right), \tag{6}$$

where

$$a_k = \frac{\beta_k - \beta_{k-1}}{\sqrt{2\pi}}, \; b_k = \frac{\beta_k^2 + \beta_k\beta_{k-1} + \beta_{k-1}^2}{6}, \tag{7}$$

are constant coefficients that are independent of x.

According to the definition of the CDF of the standard Gaussian distribution, a new set of upper bounds are obtained as

$$Q_{\mathrm{UB}}(x) = 1 - F_{\mathrm{LB}}(x) = \frac{1}{2} - \sum_{k=1}^{n} a_k x \exp\left(-b_k x^2\right). \tag{8}$$

We select several groups of β values to determine a_k and b_k, and get a series of new upper bounds:

$$\begin{aligned} Q_{\mathrm{UB\text{-}2}}(x) &= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{1}{6}x^2\right), \\ Q^1_{\mathrm{UB\text{-}3}}(x) &= \frac{1}{2} - \frac{1}{3\sqrt{2\pi}} x \exp\left(-\frac{1}{54}x^2\right) - \frac{2}{3\sqrt{2\pi}} x \exp\left(-\frac{13}{54}x^2\right), \\ Q^2_{\mathrm{UB\text{-}3}}(x) &= \frac{1}{2} - \frac{1}{2\sqrt{2\pi}} x \exp\left(-\frac{1}{24}x^2\right) - \frac{1}{2\sqrt{2\pi}} x \exp\left(-\frac{7}{24}x^2\right), \\ Q^3_{\mathrm{UB\text{-}3}}(x) &= \frac{1}{2} - \frac{2}{3\sqrt{2\pi}} x \exp\left(-\frac{2}{27}x^2\right) - \frac{1}{3\sqrt{2\pi}} x \exp\left(-\frac{19}{54}x^2\right), \\ Q_{\mathrm{UB\text{-}4}}(x) &= \frac{1}{2} - \frac{1}{3\sqrt{2\pi}} x \exp\left(-\frac{1}{54}x^2\right) - \frac{1}{3\sqrt{2\pi}} x \exp\left(-\frac{7}{54}x^2\right) \\ &\quad - \frac{1}{3\sqrt{2\pi}} x \exp\left(-\frac{19}{54}x^2\right). \end{aligned} \tag{9}$$

The lower bounds on the CDF are tight for small argument values only and they approach the exact value with increasing n. Fig. 1 shows that our 2-term, 3-terms and 4-terms bounds are all tighter than the upper bounds of Chiani's and Fu's for small argument values. The 2-term bound $Q_{\mathrm{UB\text{-}2}}(x)$ turns out to be invertible, as will be shown in Section 2. Considering accuracy and conciseness of the expressions, $Q^3_{\mathrm{UB\text{-}3}}(x)$ is chosen for small argument values, as shown in Fig. 2.

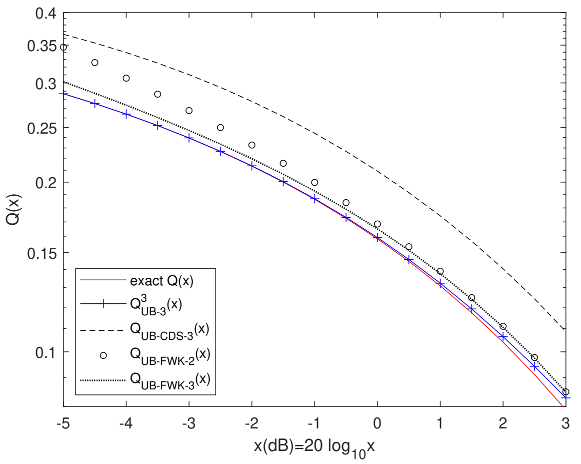

Fig 2 *The new upper bound on the Gaussian Q-function for small argument values.*

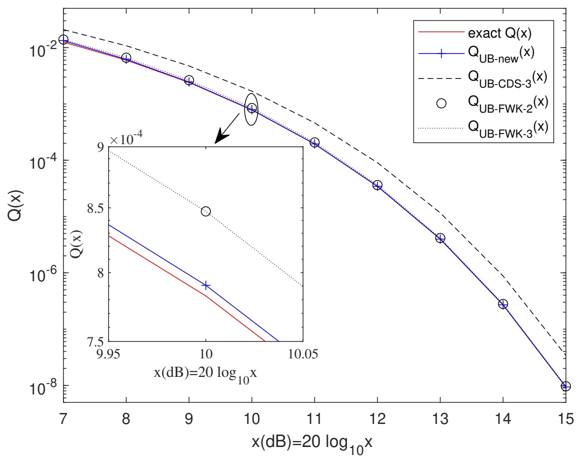

Fig 3 *The new upper bound on the Gaussian Q-function for large argument values.*

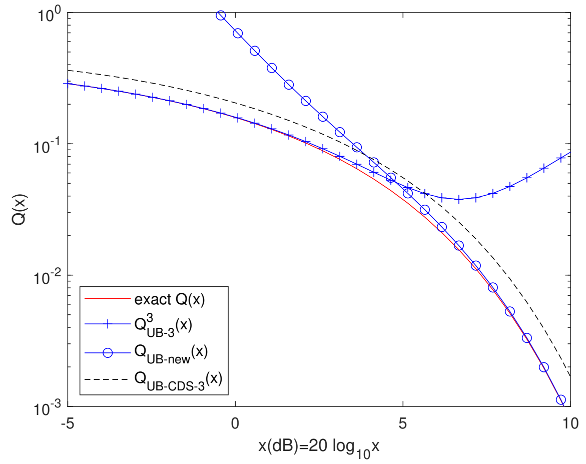

Fig 4 *The piece-wise upper bound on the Gaussian Q-function.*

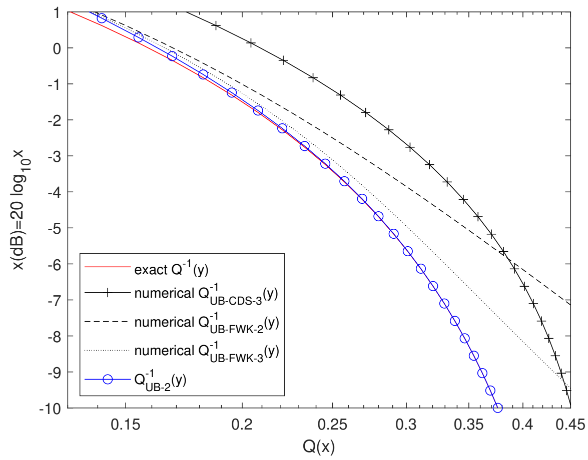

Fig 5 *The invertible upper bound on the Gaussian Q-function.*

Using integration by parts, the following expression for the Gaussian Q-function in [6] is obtained

$$Q(x) = \frac{1}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2}\right)\exp\left(-\frac{x^2}{2}\right) + \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{3}{t^4}\exp\left(-\frac{t^2}{2}\right)dt. \tag{10}$$

Then, applying integration by parts on the integral part again, (10) becomes

$$Q(x) = \frac{1}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2} + \frac{3}{x^4}\right)\exp\left(-\frac{x^2}{2}\right) - \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{15}{t^6}\exp\left(-\frac{t^2}{2}\right)dt. \tag{11}$$

By omitting the last term in (11), we obtain a new upper bound on the Gaussian Q-function as

$$Q_{\text{UB-new}}(x) = \frac{1}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2} + \frac{3}{x^4}\right)\exp\left(-\frac{x^2}{2}\right). \tag{12}$$

Fig. 3 shows that our upper bound is tighter than the upper bounds of Chiani's and Fu's for large argument values.

$Q^3_{\text{UB-3}}(x)$ is tighter than the upper bounds of Chiani's and Fu's for small argument values while $Q_{\text{UB-new}}(x)$ is tighter for large argument values. So we combine the two upper bounds into a piece-wise function, which is written as:

$$Q_{\text{UB}}(x) = \begin{cases} \frac{1}{2} - \frac{2}{3\sqrt{2\pi}}x\exp\left(-\frac{2}{27}x^2\right) - \frac{1}{3\sqrt{2\pi}}x\exp\left(-\frac{19}{54}x^2\right), & x \le 4.8 \\ \frac{1}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2} + \frac{3}{x^4}\right)\exp\left(-\frac{x^2}{2}\right), & x > 4.8 \end{cases} \tag{13}$$

The intersection of the $Q^3_{\text{UB-3}}(x)$ and $Q_{\text{UB-new}}(x)$ is at $x = 1.7378 = 4.8$ dB. Fig. 4 shows that the piece-wise upper bound is tighter than the Chiani bound $Q_{\text{UB-CDS-3}}(x)$ for both small and large argument values.

Inverse function: In some applications, the expression for $Q(x)$ needs to be inverted, e.g. to determine the SNR required for a certain error probability. Therefore, a tight invertible bound for the Gaussian Q-function is useful. The 2-term upper bound $Q_{\text{UB-2}}(x)$ in (9) is invertible. Its closed-form inverse function is given by

$$x = Q^{-1}_{\text{UB-2}}(y) = \sqrt{-3W\left(-\frac{\pi}{6}(2y-1)^2\right)}, \tag{14}$$

where the Lambert W function $W(z)$ is defined as the inverse function of [8, eq.(1.5)]

$$W(z)e^{W(z)} = z. \tag{15}$$

Fig. 5 shows the accuracy of $Q^{-1}_{\text{UB-2}}(y)$, whereas the other bounds shown in comparison are not invertible.

Symbol error probability analysis: SEP analysis plays an important role in performance analysis of communication systems. The SEP expressions for various digital modulation techniques, e.g. quadrature phase shift keying (QPSK), differentially encoded QPSK (DE-QPSK) and M-ary pulse amplitude modulation (MPAM), over the additive white Gaussian noise (AWGN) channel are commonly expressed using the Gaussian Q-function, and are further extended to derive SEP expressions over fading channels.

The SEP of QPSK is given as [6, eq.(58)]:

$$P^{\text{AWGN}}_{\text{QPSK}} = 2Q\left(\sqrt{\gamma}\right) - Q^2\left(\sqrt{\gamma}\right). \tag{16}$$

The SEP of DE-QPSK (coherent detection) is given as [6, eq.(59)] and [9, eq.(11)]:

$$P^{\text{AWGN}}_{\text{DE-QPSK}} = 4Q\left(\sqrt{\gamma}\right) - 8Q^2\left(\sqrt{\gamma}\right) + 8Q^3\left(\sqrt{\gamma}\right) - 4Q^4\left(\sqrt{\gamma}\right). \tag{17}$$

Fig. 6 and Fig. 7 show that the SEP performance of QPSK and DE-QPSK over AWGN channel using the new piece-wise upper bound is tighter

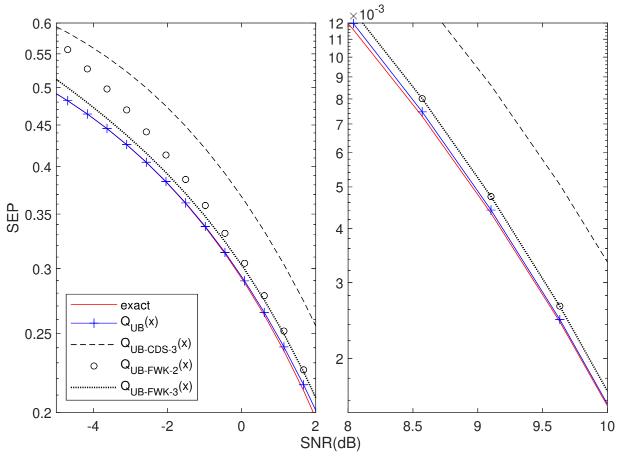

Fig 6 *Upper bounds on the SEP of QPSK over AWGN.*

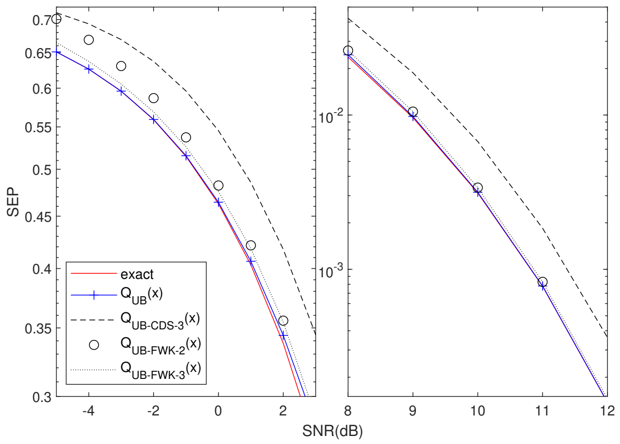

Fig 7 *Upper bounds on the SEP of DE-QPSK over AWGN.*

than the upper bounds of Chiani's and Fu's, except for the value near the intersection of the piece-wise function. Therefore, our upper bound is very suitable for high or low SNR, for example, when the characteristic signal itself is really weak or when the SNR is reduced due to the strong noise interference.

The average SEP of MPAM signals over Nakagami-m fading can be derived as [10, eq.(12)]:

$$P_e = \frac{40\,(M-1)\,/\ln 10}{M\sqrt{2\pi\sigma^2}} \int_0^\infty \frac{Q\,(t)}{t} e^{-\frac{\left(20\lg t + 10\lg\frac{M^2-1}{6} - \mu\right)^2}{2\sigma^2}} dt \qquad (18)$$

with μ and σ being the logarithmic mean and logarithmic standard deviation, respectively. In the comparison, $\Gamma = E\{\gamma\} = e^{\frac{\ln 10}{10}\mu + \frac{\ln^2 10}{200}\sigma^2}$. Fig. 8 shows that the average SEP performance of MPAM over Nakagami-m fading used by our new piece-wise upper bound is also tighter than the upper bounds of Chiani's and Fu's except for the value near the intersection of the piece-wise function. In this case, the new upper bound fits a large or small Γ value.

Conclusion: We obtain a set of new exponential upper bounds on the Gaussian Q-function by the Jensen's inequality and integration by parts. A piece-wise upper bound is obtained combining the two. Besides, we also obtain an invertible upper bound for small argument values. Applications of the piece-wise upper bound to the SEP of various digital modulation techniques over AWGN and Nakagami-m fading show the tightness of the bounds. The upper bounds together with lower bounds, e.g. those in [4, 5], alone can show the tightness of the bounds, without having to compare with exact Gaussian Q-function values.

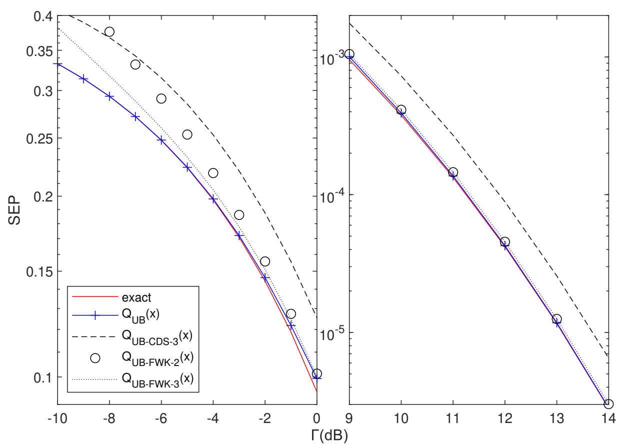

Fig 8 *Upper bounds on the SEP of MPAM over Nakagami-m fading (M = 2, σ = 2).*

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