A Bivariate Spectral Linear Partition Method for Solving Nonlinear Evolution Equations

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This work develops a method for solving nonlinear evolution equations. The method, termed a bivariate spectral linear partition method, BSLPM, combines the Chebyshev spectral collocation method, bivariate Lagrange interpolation, and a linear partition technique as an underlying linearization method. It is developed for an nth order nonlinear differential equation and then used to solve three known evolution problems. The results are compared with known exact solutions from literature. The method's applicability, reliability, and accuracy are confirmed by the congruence between the numerical and exact solutions. Tables, error graphs, and convergence graphs were generated using MATLAB (R2015a), to confirm the order of accuracy of the method and verify its convergence. The performance of the method is also observed against other methods performing well in these types of differential equations and is found to be comparable in terms of accuracy. The proposed method is also efficient as it uses minimal computation time.

1. INTRODUCTION

To this day, researchers are still working tirelessly to develop new robust numerical techniques that may best solve complex nonlinear partial differential equations. The motivation behind this is the fact that many of these nonlinear partial differential equations arising in numerous fields of mathematics, chemistry, engineering, physics, and biology, model real-life phenomenon and have many useful applications, hence finding their solutions forms an integral part of understanding their dynamics and behavior. In this paper, we present an alternative algorithm for solving nonlinear evolution equations.

Several numerical and analytical algorithms have been implemented before, to find solutions to these equations. Spectral methods have gained considerable attention over the years, particularly because of their computational efficiency and the ease with which they can be applied to variable coefficient and nonlinear differential equations [1, 2]. Previously developed numerical methods presented some pitfalls that spectral methods address. For example, analytical methods like homotopy analysis methods and homotopy perturbation methods use a series approach and their accuracy is determined by the number of terms used in the series expansion [3, 4]. However, complex nonlinear partial differential equations may require a very large number of terms to achieve accuracy and this may be tedious to handle and hence expanding room for making errors in the analysis. Moreover, a large number of terms consumes a lot of computer memory and computational time, hence making the problem cumbersome for software to solve. Spectral methods, on the other hand, do not use a series approach and only require relatively fewer grid points to achieve accuracy [5]. Besides homotopy methods, numerical methods like finite difference methods have been used to solve nonlinear partial differential equations and have been shown to have some limitations as well. These methods require many grid points to converge to an exact solution, hence demanding considerable computational time and memory [6, 7]. Spectral methods, on the other hand, have been shown to use comparatively few grid points which significantly reduces computational costs and improves accuracy.

Spectral methods have also been used in conjunction with other numerical methods to solve different classes of differ-

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ential equations. For example, a technique called the finite difference-spectral method combines the finite difference method in time and the spectral method in space to solve differential equations [8]. Another hybrid technique termed the spectral-homotopy analysis method uses a perturbation technique and defines the auxiliary linear operator in terms of the Chebyshev spectral collocation differentiation matrix [9]. These, however, presented some shortcomings as far as accuracy and convergence are concerned. This is due to the fact they discretize derivatives in space using spectral methods while discretizing derivatives in time using another numerical technique, hence, the development of pseudo-spectral methods that discretize both space and time derivatives using spectral methods. These methods are the bivariate spectral quasilinearization method (BSQLM), the bivariate spectral local linearization method (BSLLM), and the bivariate spectral relaxation method (BSRM).

The bivariate spectral quasilinearization method, BSQLM, was introduced by Motsa (2014) to solve nonlinear evolution equations namely; the modified KdV-Burger's equation, the highly nonlinear modified KdV equation, Fisher's equation, Burger's-Fisher equation, Burger's-Huxley equation, and the Fitzhugh-Nagumo equation [10].

In recent development of new linearization techniques, the bivariate spectral relaxation method, BSRM, has been used to solve nonlinear systems of steady nonsimilar boundary layer partial equations [11]. The bivariate spectral local linearization method has been applied to solve systems of non-similar boundary layer equations [11]. The method was also employed for unsteady magnetohydrodynamic micropolar-nanofluids with homogeneous-heterogeneous chemical reactions over a stretching surface [12]. The BSLLM also solved an unsteady two-dimensional boundary layer flow with heat and mass transfer, in another study [13]

Due to the overall advantages bivariate Chebyshev spectral methods have, as observed in the literature, this work employs the same approach with a new linearization technique to develop a method to solve nonlinear partial differential equations. This method brings forth several advantages over the linearization techniques mentioned above. Firstly, it is elementary in as far as implementation is concerned and generalizing it for solving other partial differential equations is straightforward. Additionally, it does not use derivatives, hence it is immune from drawbacks presented by other linearization methods like the quasilinearization method, QLM, which include the fact that the function and its derivatives must be continuous and calculable, otherwise, the method will not converge and hence fails. With fewer derivatives involved in the development of the scheme with this novel method, it then uses fewer functions evaluation than previously used methods. Additionally, the iterative schemes are simply derived directly from the governing equation, without additional manipulation, which guarantees the scheme's consistency.

The applicability, accuracy, and reliability of the proposed BSLPM are confirmed by solving three evolution problems sourced from literature and their results are compared against known exact solutions that have been reported in the scientific literature. It is observed that the method achieves high accuracy with relatively fewer spatial grid points. It also converges fast to the exact solution and is computationally efficient, with simulations completed in fractions of a second in all cases studied. Tables are generated to show the order of accuracy of the method and the time taken to compute the solutions. It is observed that the error decreases with an increasing number of grid points. Error graphs and convergence graphs showing the excellent agreement of the exact and analytical solutions for all the nonlinear evolution equations and verifying the convergence of the method are also presented.

The paper is organized as follows. In Section 2, we introduce the BSLPM algorithm for a general nonlinear evolution PDE. In Section 3, we describe the application of the BSLPM to selected test problems. The numerical simulations and results are presented in Section 4. Finally, we conclude in Section 5.

2. BIVARIATE SPECTRAL LINEAR PARTITION METHOD

In this section, we introduce the Bivariate Spectral Linear Partition Method (BSLPM) for finding solutions to non-linear evolution PDEs. Without loss of generality, we consider nonlinear PDEs of the form ;

$$\frac{\partial f}{\partial \tau} = H\left(f, \frac{\partial f}{\partial \eta}, \frac{\partial^2 f}{\partial \eta^2}, ..., \frac{\partial^n f}{\partial \eta^n}\right), \tag{1}$$

with physical regions $\tau \in [0, T]$, $\eta \in [a, b]$, where n, is the order of differentiation, $f(\eta, \tau)$ is the required solution and H is the non-linear operator which contains all the spatial derivatives of f. The choice of grid points, Chebyshev-Gauss Lobatto grid points, to be used in the solution procedure are defined in the interval $[-1, 1]$, therefore, it is convenient to transform the time interval $\tau \in [0, T]$ and space region $\eta \in [a, b]$ to $[-1, 1]$ using the linear transformations $\tau = \frac{T(\xi + 1)}{2}$ and $\eta = \frac{(b - a)\zeta}{2} + \frac{b + a}{2}$, respectively.

After the transformation, Eq. (1) can be expressed as

$$\frac{\partial f}{\partial \xi} = H\left(f, \frac{\partial f}{\partial \zeta}, \frac{\partial^2 f}{\partial \zeta^2}, ..., \frac{\partial^n f}{\partial \zeta^n}\right). \tag{2}$$

The solution procedure, as proposed by [10], assumes that the solution can be approximated by a bivariate Lagrange interpolation polynomial of the form

$$f(\zeta, \xi) \approx \sum_{i=0}^{N_\zeta} \sum_{j=0}^{N_\xi} f(\zeta_i, \xi_j) L_i(\zeta) L_j(\xi), \tag{3}$$

which interpolates $f(\zeta, \xi)$ at selected points in both the ζ and ξ directions defined by

$$\{\zeta_i\} = \left\{\cos\left(\frac{\pi i}{N_\zeta}\right)\right\}_{i=0}^{N_\zeta}, \qquad \{\xi_j\} = \left\{\cos\left(\frac{\pi j}{N_\xi}\right)\right\}_{j=0}^{N_\xi}, \tag{4}$$

where the function $L_i(\zeta)$ is the well-known characteristic lagrange cardinal polynomial based on the Chebyshev-Gauss Lobatto points [14]

$$L_i(\zeta) = \prod_{\substack{i=0 \\ i \neq k}}^{N_\zeta} \frac{\zeta - \zeta_k}{\zeta_i - \zeta_k}, \tag{5}$$

which obeys the Kronecker delta equation

$$L_i(\zeta_k) = \delta_{i,k} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}. \tag{6}$$

$L_j(\xi)$ is defined in a similar way.
To express equation Eq. (2) in linear form, we first split it into its linear and non-linear components such that the governing equation is written in the form;

$$P[f, f', ..., f^{(n)}] + Q[f, f', ..., f^{(n)}] - \dot{f} = 0. \tag{7}$$

The prime and dot denote the space and time derivatives, respectively. P is a linear operator and Q is the non-linear operator.

To simplify the non-linear component, Q, we consider two cases of the linear partition method implemented as follows;

Case 1: In the first case, all linear terms of the governing differential equation, including the time derivative, are evaluated at the current iteration, $(r+1)$, while all non-linear terms at the previous (r) to obtain

$$\dot{f}_{r+1} = P[f_{r+1}, f'_{r+1}, f''_{r+1}] + Q[f_r, f'_r, f''_r]. \tag{8}$$

Case 2: The second case prioritizes higher order derivatives of non-linear terms and we define the following rules;

1. In a case where the nonlinear term is a product of derivatives, we take the higher order derivative in the term and evaluate it at the current iteration, $(r+1)$, and the remaining at the previous iteration, (r). For example, if the nonlinear product is $f^{(n-1)}(\zeta)f^{(n)}(\zeta)$ where n is the order of differentiation, we will obtain $f_r^{(n-1)}(\zeta)f_{r+1}^{(n)}(\zeta)$.

2. If the term is a product of derivatives with the unknown function, we again take the higher order derivative and evaluate it at the current iteration and the remaining function at the previous. For example, if we have $f(\zeta)f^{(n)}(\zeta)$ we will obtain $f_r(\zeta)f_{r+1}^{(n)}(\zeta)$.

3. If the nonlinear term is just a product of the unknown function, we simply evaluate it at the previous iteration, i.e., if we have $f^2(\zeta)$ we simply set $f_r^2(\zeta)$.

4. Lastly, if the derivative function has a power, i.e $[f(\zeta)f^{(n)}(\zeta)]^k$ and $[f^{(n)}(\zeta)]^k$, where k is a constant, the above second and third approaches are taken respectively, i.e, $[f_r(\zeta)f_{r+1}^{(n)}(\zeta)]^k$ and $[f_r^{(n)}(\zeta)]^k$, respectively.

Typically, a general linearized scheme of Eq. (8), will be as follows

$$\phi_{n,r} f_{r+1}^{(n)} + \dots + \phi_{2,r} f_{r+1}'' + \phi_{1,r} f_{r+1}' - \dot{f}_{r+1} = R_r[f_r, f_r', \dots, f_r^{(n)}]. \tag{9}$$

The process of collocation involves evaluating the differential equation at the collocation points (ζ_i, ξ_j) for $i = 0, 1, \dots, N_\zeta$ and $j = 0, 1, \dots, N_\xi$. In the collocation process, we first apply the collocation process in ξ to obtain an ODE which is then solved by collocating in ζ.

The value of the time derivatives at the Chebyshev Gauss Lobatto points ζ_i, ξ_j for $j = 0, 1, \dots, N_\xi$ are computed as

$$
\begin{aligned}
\left. \frac{\partial f}{\partial \xi} \right|_{\zeta=\zeta_i, \xi=\xi_j} &= \sum_{p=0}^{N_\zeta} \sum_{q=0}^{N_\xi} f(\zeta_p, \xi_q) L_p(\zeta_i) \frac{dL_q(\xi_j)}{d\xi}, \\
&= \sum_{q=0}^{N_\xi} f(\zeta_i, \xi_q) d_{j,q}, \\
&= \sum_{q=0}^{N_\xi} d_{j,q} f(\zeta_i, \xi_q),
\end{aligned} \tag{10}
$$

since for $i = p$, $\sum_{p=0}^{N_\zeta} L_p(\zeta_i) = 1$ and 0 otherwise, where $d_{j,q} = \dfrac{dL_q(\xi_j)}{d\xi}$ is the jth and q^{th} entry of the standard first derivative Chebyshev differentiation matrix of size $(N_\xi + 1) \times (N_\xi + 1)$, for $j, q = 0, 1, \dots, N_\xi$, as defined in [15].

The value of the space derivatives at the Chebyshev-Gauss Lobatto points (ζ_i, ξ_j) for $i = 0, 1, \dots, N_\zeta$ are computed as

$$
\begin{aligned}
\left. \frac{\partial f}{\partial \zeta} \right|_{\zeta=\zeta_i, \xi=\xi_j} &= \sum_{p=0}^{N_\zeta} \sum_{q=0}^{N_\xi} f(\zeta_p, \xi_q) L_q(\xi_j) \frac{dL_p(\zeta_i)}{d\zeta}, \\
&= \sum_{p=0}^{N_\zeta} f(\zeta_p, \xi_j) D_{i,p}, \\
&= \sum_{p=0}^{N_\zeta} D_{i,p} f(\zeta_p, \xi_j), = \mathbf{DF}_j
\end{aligned} \tag{11}
$$

where $\mathbf{D}$ is the standard first derivative Chebyshev matrix of size $(N_\zeta + 1) \times (N_\zeta + 1)$ with its ith and p^{th} entries given by $D_{i,p} = \dfrac{dL_p(\zeta_i)}{d\zeta}$, for $i, p = 0, 1, \dots, N_\zeta$.

Similarly, for the higher n^{th} order derivative, we have

$$
\begin{aligned}
\left. \frac{\partial^n f}{\partial \zeta^n} \right|_{\zeta=\zeta_i, \xi=\xi_j} &= \sum_{p=0}^{N_\zeta} f(\zeta_p, \xi_j) D_{i,p}^n, \\
&= \mathbf{D}^n \mathbf{F}_j,
\end{aligned} \tag{12}
$$

for $i = 0, 1, \dots, N_\zeta$, where the vector $\mathbf{F}_j$ is defined as

$$\mathbf{F}_j = [f_j(\zeta_0), f_j(\zeta_1), \dots, f_j(\zeta_{N_\zeta})]^T, \tag{13}$$

and the superscript T denotes matrix transpose.

Hence applying spectral collocation to equation Eq. (9) by substituting these derivatives on the linearized scheme and imposing the initial condition we obtain

$$\mathbf{\Phi}_{n,r} \mathbf{D}^{(n)} \mathbf{F}_{r+1,i} + \dots + \mathbf{\Phi}_{1,r} \mathbf{DF}_{r+1,i} + \mathbf{\Phi}_{0,r} \mathbf{F}_{r+1,i} - \sum_{j=0}^{N_\xi - 1} d_{ij} \mathbf{F}_{r+1,j} = R_r[\mathbf{F}_{r,i}, \mathbf{F}_{r,i}', \dots, \mathbf{F}_{r,i}^{(n)}] + d_{i,N_\xi} \mathbf{F}_{N_\xi}, \tag{14}$$

where

$$\mathbf{\Phi}_{n,r} = \begin{bmatrix} \phi_{n,r}(\zeta_0, \xi_i) & & & & & \\ & \phi_{n,r}(\zeta_1, \xi_i) & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \phi_{n,r}(\zeta_{N_\zeta}, \xi_i) \end{bmatrix}.$$ (15)

Eq. (14) can be expressed in an $N_\xi(N_\zeta + 1) \times N_\xi(N_\zeta + 1)$ matrix system as shown below;

$$\begin{bmatrix} A_{0,0} & A_{0,1} & \cdot & \cdot & \cdot & A_{0,N_\xi-1} \\ A_{1,0} & A_{1,1} & \cdot & \cdot & \cdot & A_{1,N_\xi-1} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ A_{N_\xi-1,0} & A_{N_\xi-1,0} & \cdot & \cdot & \cdot & A_{N_\xi-1,N_\xi-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{F}_{N_\xi-1} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_0 \\ \mathbf{R}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{R}_, \end{bmatrix}$$ (16)

where

$$A_{i,i} = \mathbf{\Phi}_{n,r}^{(i)} \mathbf{D}^{(n)} + ... + \mathbf{\Phi}_{1,r}^{(i)} \mathbf{D} + \mathbf{\Phi}_{0,r}^{(i)} - d_{i,i} \mathbf{I}$$ (17)

$$A_{i,j} = -d_{i,j} \mathbf{I} \quad \text{when } i \neq j$$ (18)

and $\mathbf{I}$ is the identity matrix of size $(N_\zeta + 1) \times (N_\zeta + 1)$. Solving equation Eq. (16) gives the approximate solution of $f(\zeta, \xi)$.

3. NUMERICAL EXPERIMENTS

We apply the BSLPM algorithm to well-known nonlinear evolution equations in this subsection.

A. Burger's-Fisher equation

We consider the well-known Burgers-Fisher equation, a combined form of Fisher's and Burger's equations [16–18]. as

$$\frac{\partial f}{\partial \xi} = \frac{\partial^2 f}{\partial \zeta^2} - \alpha f \frac{\partial f}{\partial \zeta} + \beta f - \beta f^{1+\delta}.$$ (19)

When $\alpha = \beta = \delta = 1$, [19] we have

$$\frac{\partial f}{\partial \xi} = \frac{\partial^2 f}{\partial \zeta^2} - f \frac{\partial f}{\partial \zeta} + f - f^2, \quad 0 \leq \zeta \leq 1, \quad 0 \leq \xi \leq 1,$$ (20)

subject to initial condition

$$f(\zeta, 0) = \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{\zeta}{4}\right),$$ (21)

and boundary conditions

$$f(0, \xi) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{5\xi}{8}\right),$$ (22)

$$f(1, \xi) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{5\xi}{8} - \frac{5}{4}\right).$$ (23)

The exact solution is given by

$$f(\zeta, \xi) = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{5\xi}{8} - \frac{\zeta}{4}\right).$$ (24)

The linear and nonlinear operators are chosen as

$$P(f) = f'' + f - \dot{f}, \quad Q = -ff' - f^2. \tag{25}$$

Case 1: The linearized scheme obtained is

$$f''_{r+1} + f_{r+1} - \dot{f}_{r+1} = f_r^2 + f_r f'_r, \tag{26}$$

and applying spectral collocation and imposing the initial condition

$$f(\zeta, 0) = \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{\zeta}{4}\right), \tag{27}$$

we obtain

$$\mathbf{D}^2 \mathbf{F}_{r+1,i} + \mathbf{F}_{r+1,i} - \sum_{j=0}^{N_\xi - 1} d_{ij} \mathbf{F}_{r+1,j} = \mathbf{R}_r + d_{iN_\xi} F_{N_\xi}. \tag{28}$$

Case 2: Applying the second case of the linear partition method, we obtain the scheme below

$$f''_{r+1} - f_r f'_{r+1} + f_{r+1} - \dot{f}_{r+1} = f_r^2. \tag{29}$$

Applying spectral collocation and imposing the initial condition, we obtain

$$\mathbf{D}^2 \mathbf{F}_{r+1,i} + \mathbf{\Phi}_{1,r} \mathbf{D} \mathbf{F}_{r+1,i} + \mathbf{F}_{r+1,i} - \sum_{j=0}^{N_\xi - 1} d_{ij} \mathbf{F}_{r+1,j} = R_r[\mathbf{F}_{r,i}, \mathbf{F}'_{r,i}, ..., \mathbf{F}^{(n)}_{r,i}] + d_{iN_\xi} \mathbf{F}_{N_\xi}, \tag{30}$$

where

$$R_r = \mathbf{F}^2_{r,i}, \tag{31}$$

and

$$\mathbf{\Phi}_{1,r} = -\mathbf{F}_{r,i}. \tag{32}$$

Equations Eq. (28) and Eq. (30) can be expressed in $N_\xi(N_\zeta + 1) \times N_\xi(N_\zeta + 1)$ matrix systems as shown in Eq. (16). The procedure remains the same in all the following examples.

B. Burger's-Huxley equation

We also consider the Burgers–Huxley [20] equation which also came into existence due to the combined effects of the Burgers equation and the Huxley equation.

$$\frac{\partial f}{\partial \xi} + f \frac{\partial f}{\partial \zeta} = \frac{\partial^2 f}{\partial \zeta^2} + f(1 - f)(f - 0.1), \tag{33}$$

subject to the initial condition

$$f(\zeta, 0) = \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{\zeta}{2}\right), \tag{34}$$

and boundary conditions

$$f(0, \xi) = \frac{1}{2} - \frac{1}{2} \tanh\left[\frac{1}{2}(-0.9\xi)\right], \quad f(1, \xi) = \frac{1}{2} - \frac{1}{2} \tanh\left[\frac{1}{2}(1 - 0.9\xi)\right]. \tag{35}$$

The exact solution is given by

$$f(\zeta, \xi) = \frac{1}{2} - \frac{1}{2} \tanh\left[\frac{1}{2}(\zeta - 0.9\xi)\right]. \tag{36}$$

The linear operators P and nonlinear operator Q are chosen as

$$P(f) = f'' - 0.1f - \dot{f} \quad \text{and} \quad Q(f) = -ff' + 1.1f^2 - f^3. \tag{37}$$

Case 1: The linearized scheme of the problem becomes

$$f''_{r+1} - f_r f'_r - f_r^3 + 1.1f_r^2 - 0.1f_{r+1} - \dot{f}_{r+1} = 0. \tag{38}$$

Applying spectral collocation in both ζ and ξ, and imposing the initial condition we obtain

$$\mathbf{D}^2\mathbf{F}_{r+1,i} - 0.1\mathbf{F}_{r+1,i} - \sum_{j=0}^{N_\xi-1} d_{ij}\mathbf{F}_{r+1,j} = R_r[\mathbf{F}_{r,i}, \mathbf{F}'_{r,i}, ..., \mathbf{F}^{(n)}_{r,i}] + d_{iN_\xi}\mathbf{F}_{N_\xi}, \tag{39}$$

where

$$R_r = \mathbf{F}^3_{r,i} - 1.1\mathbf{F}^2_{r,i} + \mathbf{F}_{r,i}\mathbf{F}'_{r,i}. \tag{40}$$

Case 2: The linearized scheme of the problem becomes

$$f''_{r+1} - f_r f'_{r+1} - f_r^3 + 1.1f_r^2 - 0.1f_{r+1} - \dot{f}_{r+1} = 0. \tag{41}$$

Applying spectral collocation in both ζ and ξ, and imposing the initial condition we obtain

$$\mathbf{D}^2\mathbf{F}_{r+1,i} + \mathbf{\Phi}_{1,r}\mathbf{F}_{r+1,i} - 0.1\mathbf{F}_{r+1} - \sum_{j=0}^{N_\xi-1} d_{ij}\mathbf{F}_{r+1,k} = R_r[\mathbf{F}_{r,i}, \mathbf{F}'_{r,i}, ..., \mathbf{F}^{(n)}_{r,i}] + d_{iN_\xi}\mathbf{F}_{N_\xi}, \tag{42}$$

where

$$\mathbf{\Phi}_{1,r} = -\mathbf{F}_{r,i} \tag{43}$$

$$R_r = \mathbf{F}^3_{r,i} - 1.1\mathbf{F}^2_{r,i} \tag{44}$$

C. Example 4.1

Consider the nonlinear evolution partial differential equation

$$\frac{\partial f}{\partial \xi} = \frac{\partial^2 f}{\partial \zeta^2} + f\frac{\partial f}{\partial \zeta} + \frac{1}{9}f(f^2 - 36), \tag{45}$$

subject to the initial condition

$$f(\zeta, 0) = \frac{6(e^{2\zeta} - 1)}{1 + e^\zeta + e^{2\zeta}}, \tag{46}$$

and boundary conditions

$$f(0, \xi) = 0, \qquad f(1, \xi) = \frac{6(e^2 - 1)}{1 + e^2 + e^{1+\xi}}. \tag{47}$$

The exact solution is

$$f(\zeta, \xi) = \frac{6(e^{2\zeta} - 1)}{1 + e^{2\zeta} + e^{\zeta+3\xi}}. \tag{48}$$

Case 1: The linearised scheme of the problem becomes

$$f''_{r+1} + f_r f'_r + \frac{1}{9}f_r^3 - 4f_{r+1} - \dot{f}_{r+1} = 0.$$

Applying spectral collocation in both ζ and ξ, and imposing the initial condition we obtain

$$\mathbf{D}^2\mathbf{F}_{r+1,i} - 4\mathbf{F}_{r+1,i} - \sum_{j=0}^{N_\xi-1} d_{i,j}\mathbf{F}_{r+1,j} = R_r[\mathbf{F}_{r,i}, \mathbf{F}'_{r,i}, ..., \mathbf{F}^{(n)}_{r,i}] + d_{iN_\xi}\mathbf{F}_{N_\xi}, \tag{49}$$

where

$$R_r = -\mathbf{F}_{r,i}\mathbf{F}'_{r,i} - \frac{1}{9}\mathbf{F}^3_{r,i}. \tag{50}$$

Case 2: The linearised scheme of the problem becomes

$$f''_{r+1} + f_r f'_{r+1} + \frac{1}{9}f_r^3 - 4f_{r+1} - \dot{f}_{r+1} = 0. \tag{51}$$

Applying spectral collocation in both ζ and ξ, and imposing the initial condition we obtain

$$\mathbf{D}^2\mathbf{F}_{r+1,i} + \mathbf{\Phi}_{1,r}\mathbf{D}\mathbf{F}_{r+1,i} - 4\mathbf{F}_{r+1,i} - \sum_{j=0}^{N_\xi-1} d_{i,j}\mathbf{F}_{r+1,j} = R_{r[\mathbf{F}_{r,i}, \mathbf{F}'_{r,i}, ..., \mathbf{F}^{(n)}_{r,i}]} + d_{iN_\xi}\mathbf{F}_{N_\xi}, \tag{52}$$

where

$$\Phi_{1,r} = \mathbf{F}_{r,i}, \tag{53}$$

$$R_r = -\frac{1}{9}\mathbf{F}_{r,i}^3. \tag{54}$$

In the next section, we present the numerical solutions obtained using the BSLPM algorithm .

4. RESULTS AND DISCUSSION

In this section, we present and discuss results obtained when solving the three nonlinear evolution partial differential equations in the previous chapter. We present, in particular, graphs of the approximate solutions of the BSLPM vs exact solutions as sourced from literature. To determine the level of accuracy of the BSLPM approximate solution at a certain time level, in comparison with the exact solutions of the problems reported in the literature, we report the infinite error norm which is defined by

$$\text{Error norm} = ||f(\zeta, \xi_j) - \tilde{f}(\zeta, \xi_j)||_\infty , \tag{55}$$

where $f(\zeta, \xi_j)$ and $\tilde{f}(\zeta, \xi_j)$ are the exact and approximate solutions, respectively. It was found that sufficient accuracy was achieved when $N_\zeta = 10$ and $N_\xi = 20$ in all numerical simulations. We present tables showing the errors of the BSLPM and then compare the general performance of the BSLPM and MDBSLPM through convergence graphs and maximum error analysis.

A. Exact Solution versus Approximate Solution

[Case 1]   [Case 2]
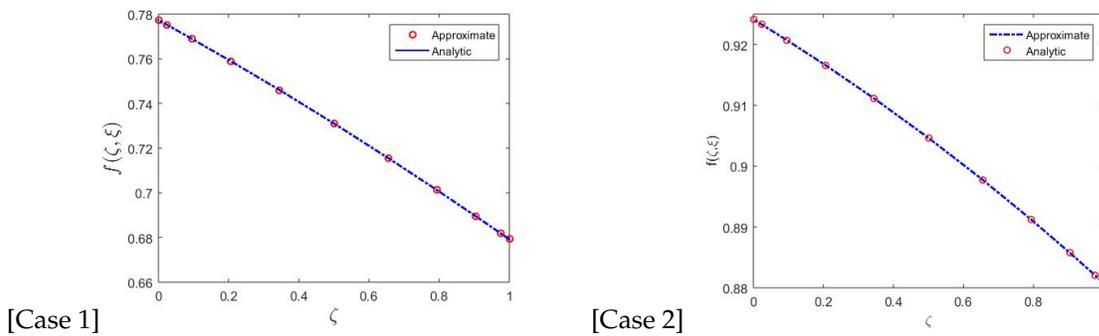
Fig. 1. Approximate vs Exact solutions of the Burgers-Fisher equation .

[Case 1]   [Case 2]
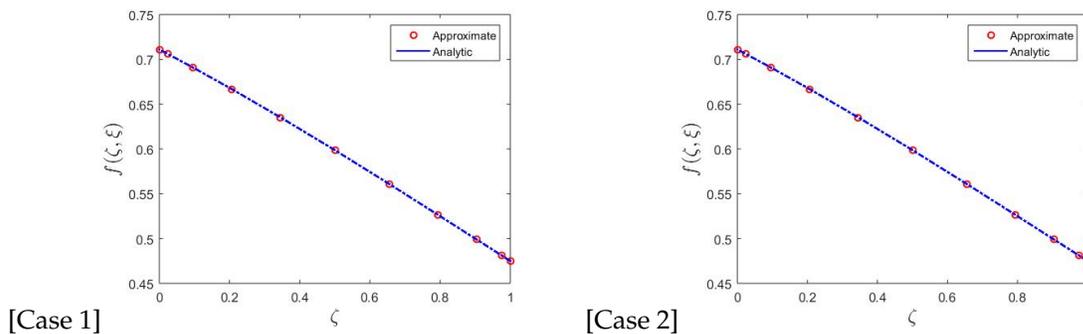
Fig. 2. Approximate vs Exact solutions of the Burgers-Huxley equation .

Figures 1, 2, 3 show a comparison of the approximate solutions and exact solutions of the numerical problems solved. The approximate solutions are in excellent agreement with the analytical solutions, and this affirms the accuracy of the algorithm presented in this study.

B. Maximum Errors
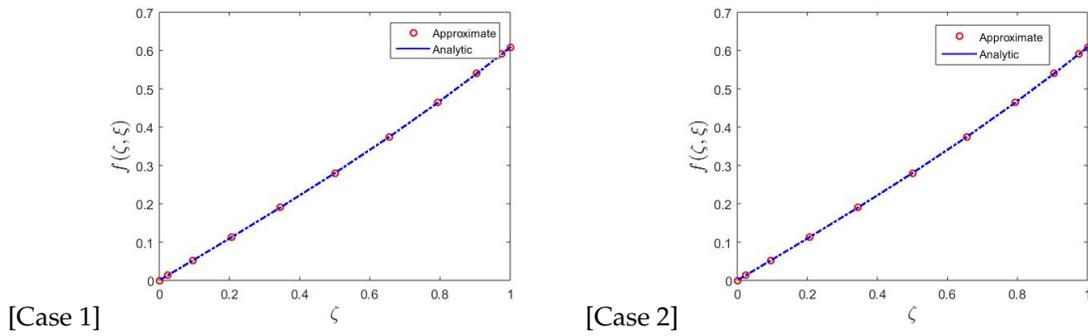
Fig. 3. Approximate vs Exact solutions of Example 4.1.

Table 1. BSLPM Case 1: Maximum errors for Burgers-Fisher equation for $N_{\tilde{\xi}} = 20$.

ξ / N_ζ	4	6	8	10
0.1	$1.126e - 07$	$1.357e - 10$	$1.357e - 10$	$3.464e - 14$
0.2	$1.187e - 07$	$1.426e - 10$	$2.387e - 13$	$2.441e - 13$
0.3	$1.175e - 07$	$1.324e - 10$	$1.538e - 13$	$1.463e - 13$
0.4	$1.075e - 07$	$1.101e - 10$	$1.098e - 13$	$1.174e - 13$
0.5	$9.062e - 08$	$8.314e - 11$	$9.004e - 14$	$5.873e - 14$
0.6	$6.903e - 08$	$4.881e - 11$	$2.720e - 14$	$6.295e - 14$
0.7	$4.682e - 08$	$1.324e - 11$	$5.373e - 14$	$1.154e - 13$
0.8	$2.477e - 08$	$3.930e - 11$	$1.019e - 13$	$2.202e - 13$
0.9	$2.030e - 08$	$6.350e - 11$	$3.524e - 13$	$7.112e - 13$
1.0	$3.147e - 08$	$7.793e - 11$	$1.052e - 13$	$2.960e - 13$
$CPUTime(sec)$	0.125606	0.120672	0.158861	0.174881

Table 2. BSLPM Case 2: Maximum errors for Burgers-Fisher equation for $N_{\tilde{\xi}} = 20$.

ξ / N_ζ	4	6	8	10
0.1	$1.126e - 07$	$1.358e - 10$	$1.579e - 13$	$5.695e - 14$
0.2	$1.187e - 07$	$1.426e - 10$	$2.142e - 13$	$3.431e - 14$
0.3	$1.175e - 07$	$1.324e - 10$	$1.711e - 13$	$1.321e - 13$
0.4	$1.075e - 07$	$1.101e - 10$	$9.448e - 14$	$3.217e - 13$
0.5	$9.062e - 08$	$8.313e - 11$	$7.050e - 14$	$1.852e - 13$
0.6	$6.903e - 08$	$4.881e - 11$	$1.987e - 14$	$8.038e - 14$
0.7	$4.682e - 08$	$1.323e - 11$	$3.797e - 14$	$1.361e - 13$
0.8	$2.477e - 08$	$3.931e - 11$	$1.915e - 13$	$4.013e - 13$
0.9	$2.030e - 08$	$6.350e - 11$	$8.793e - 14$	$2.000e - 13$
1.0	$3.147e - 08$	$7.791e - 11$	$1.298e - 13$	$4.089e - 13$
$CPUTime(sec)$	0.138967	0.116474	0.119540	0.175061

Table 3. BSLPM Case 1: Maximum errors for Burgers-Huxley equation for $N_\xi = 20$.

ξ / N_ζ	4	6	8	10
0.1	$2.468e - 06$	$1.062e - 08$	$3.380e - 11$	$9.032e - 14$
0.2	$3.171e - 06$	$1.367e - 08$	$5.356e - 11$	$1.658e - 13$
0.3	$3.610e - 06$	$1.593e - 08$	$6.948e - 11$	$3.383e - 13$
0.4	$3.872e - 06$	$1.724e - 08$	$7.942e - 11$	$2.818e - 13$
0.5	$3.980e - 06$	$1.767e - 08$	$8.296e - 11$	$3.786e - 13$
0.6	$3.944e - 06$	$1.707e - 08$	$7.931e - 11$	$3.938e - 13$
0.7	$3.771e - 06$	$1.594e - 08$	$7.558e - 11$	$3.955e - 13$
0.8	$3.474e - 06$	$1.499e - 08$	$6.723e - 11$	$4.478e - 13$
0.9	$3.148e - 06$	$1.321e - 08$	$5.351e - 11$	$3.166e - 13$
1.0	$2.862e - 06$	$1.076e - 08$	$3.599e - 11$	$6.138e - 13$
$CPUTime(sec)$	0.447930	0.113786	0.167279	0.155115

Table 4. BSLPM Case 2: Maximum errors for Burgers-Huxley equation for $N_\xi = 20$.

ξ / N_ζ	4	6	8	10
0.1	$2.468e - 06$	$1.062e - 08$	$3.380e - 11$	$9.232e - 14$
0.2	$3.171e - 06$	$1.367e - 08$	$5.360e - 11$	$2.790e - 13$
0.3	$3.610e - 06$	$1.593e - 08$	$6.946e - 11$	$2.850e - 13$
0.4	$3.872e - 06$	$1.724e - 08$	$7.938e - 11$	$4.696e - 13$
0.5	$3.980e - 06$	$1.767e - 08$	$8.298e - 11$	$3.957e - 13$
0.6	$3.944e - 06$	$1.707e - 08$	$7.930e - 11$	$3.748e - 13$
0.7	$3.771e - 06$	$1.594e - 08$	$7.557e - 11$	$4.136e - 13$
0.8	$3.474e - 06$	$1.499e - 08$	$6.724e - 11$	$4.852e - 13$
0.9	$3.148e - 06$	$1.321e - 08$	$5.354e - 11$	$1.885e - 13$
1.0	$2.862e - 06$	$1.076e - 08$	$3.596e - 11$	$6.568e - 13$
$CPUTime(sec)$	0.112422	0.118699	0.140348	0.179762

**Table 5.** BSLPM Case 1: Maximum errors for Example 4.1 . for $N_{\xi} = 20$

ξ / N_{ζ}	4	6	8	10
0.1	$4.834e - 05$	$7.875e - 07$	$6.985e - 09$	$7.702e - 11$
0.2	$3.439e - 05$	$2.364e - 07$	$2.596e - 09$	$2.355e - 11$
0.3	$1.405e - 05$	$9.856e - 08$	$6.228e - 10$	$6.492e - 12$
0.4	$9.511e - 06$	$3.114e - 08$	$2.262e - 10$	$3.997e - 12$
0.5	$1.112e - 05$	$6.300e - 08$	$2.538e - 10$	$1.480e - 12$
0.6	$1.370e - 05$	$5.134e - 08$	$2.762e - 10$	$2.446e - 12$
0.7	$1.206e - 05$	$3.238e - 08$	$1.543e - 10$	$2.617e - 12$
0.8	$8.590e - 06$	$1.520e - 08$	$1.072e - 10$	$8.816e - 13$
0.9	$5.768e - 06$	$1.066e - 08$	$4.369e - 11$	$5.008e - 13$
1.0	$3.184e - 06$	$1.331e - 08$	$3.879e - 11$	$1.115e - 13$
$CPUTime(sec)$	0.112422	0.118699	0.125990	0.143381

**Table 6.** BSLPM Case 2: Maximum errors for Example 4.1 . for $N_{\xi} = 20$

ξ / N_{ζ}	4	6	8	10
0.1	$4.834e - 05$	$7.875e - 07$	$6.985e - 09$	$7.695e - 11$
0.2	$3.439e - 05$	$2.364e - 07$	$2.596e - 09$	$2.268e - 11$
0.3	$1.405e - 05$	$9.856e - 08$	$6.226e - 10$	$6.056e - 12$
0.4	$9.511e - 06$	$3.114e - 08$	$2.256e - 10$	$3.590e - 12$
0.5	$1.112e - 05$	$6.300e - 08$	$2.538e - 10$	$1.603e - 12$
0.6	$1.370e - 05$	$5.134e - 08$	$2.763e - 10$	$2.256e - 12$
0.7	$1.206e - 05$	$3.238e - 08$	$1.543e - 10$	$2.536e - 12$
0.8	$8.590e - 06$	$1.520e - 08$	$1.072e - 10$	$1.099e - 12$
0.9	$5.768e - 06$	$1.066e - 08$	$4.374e - 11$	$6.031e - 13$
1.0	$3.184e - 06$	$1.331e - 08$	$3.878e - 11$	$1.477e - 13$
$CPUTime(sec)$	0.111851	0.119919	0.133756	0.147275

Tables 1, 2, 3, 4, 5 and 6 we give the maximum errors between the exact and BSLPM results for the three evolution problems considered in this work, at $\xi \in [0.1, 1]$, and the space domain $\zeta \in [0, 1]$. The tables show the accuracy of the method, which is seen to improve with an increasing number of collocation points in the ζ direction. The method was able to produce accurate results with errors of up to order 10^{-14}, which are comparable with results obtained from [10] for similar problems. This further validates the method. We also highlight that the BSLPM is computationally efficient as it uses few grid points to achieve accuracy and takes a fraction of a second to generate results.This is seen from the computational time taken in all the problems.

C. Error Profiles
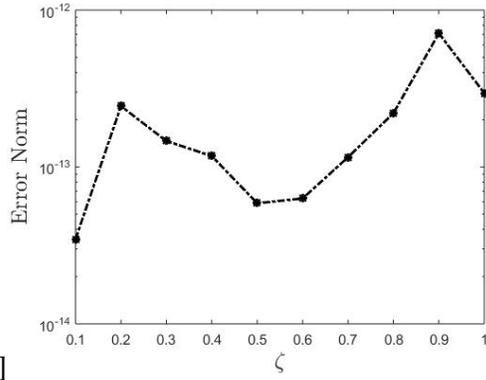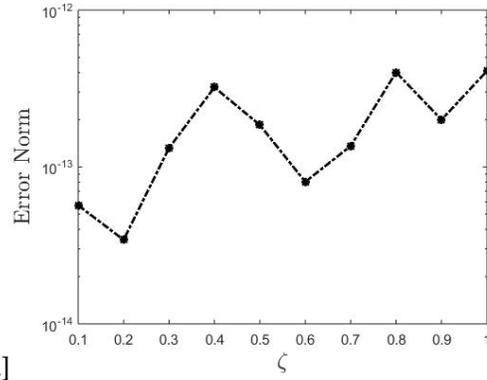
[Case 1]   [Case 2]

Fig. 4. Error profiles of Burgers-Fisher equation.
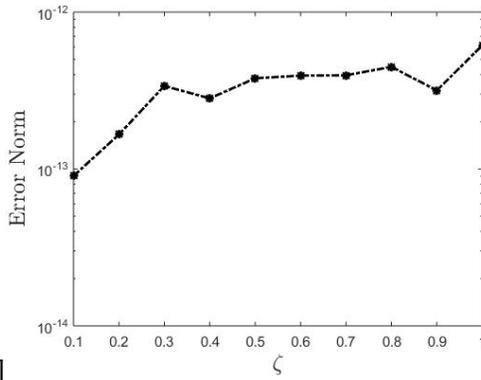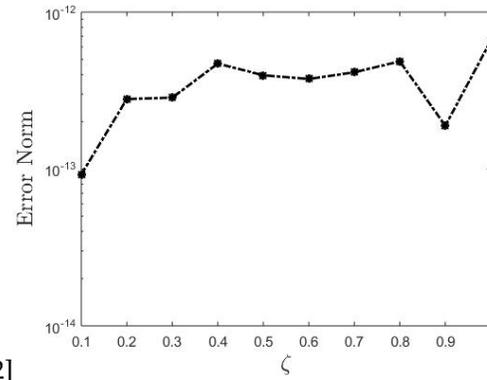
[Case 1]   [Case 2]

**Fig. 5.** Error profiles of Burgers-Huxley equation.
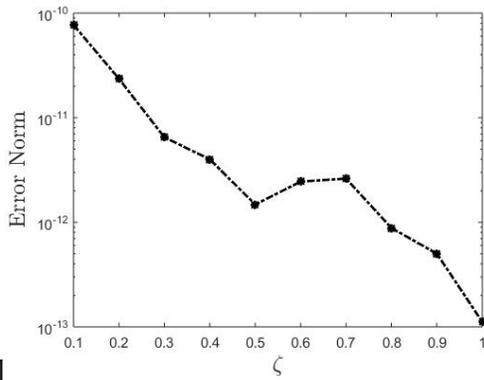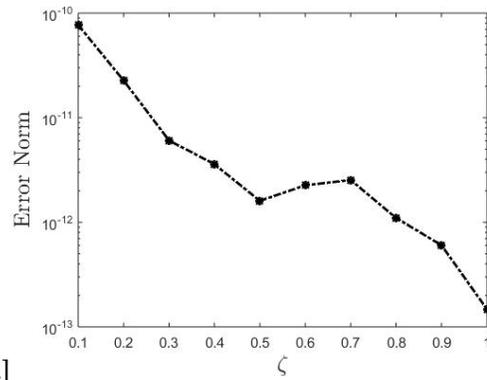
[Case 1]   [Case 2]

Fig. 6. Error profiles of Example 4.1 .

Figures 4, 5, 6 show error profiles for the three evolution equations. The errors presented here are very small hence demonstrating the accuracy of the BSLPM method over the entire domains of the nonlinear equation. The error graphs obtained are also comparable with those obtained when the BSQLM method was used to solve the problem in [10]. This further validates and affirms the accuracy of the method.

D. Convergence graphs
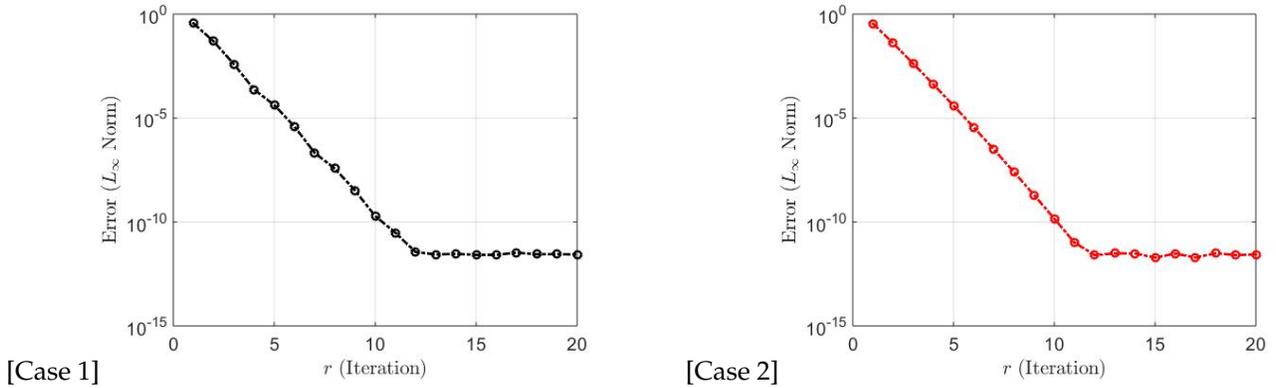
Fig. 7. Convergence graphs of the Burgers-Fisher equation.
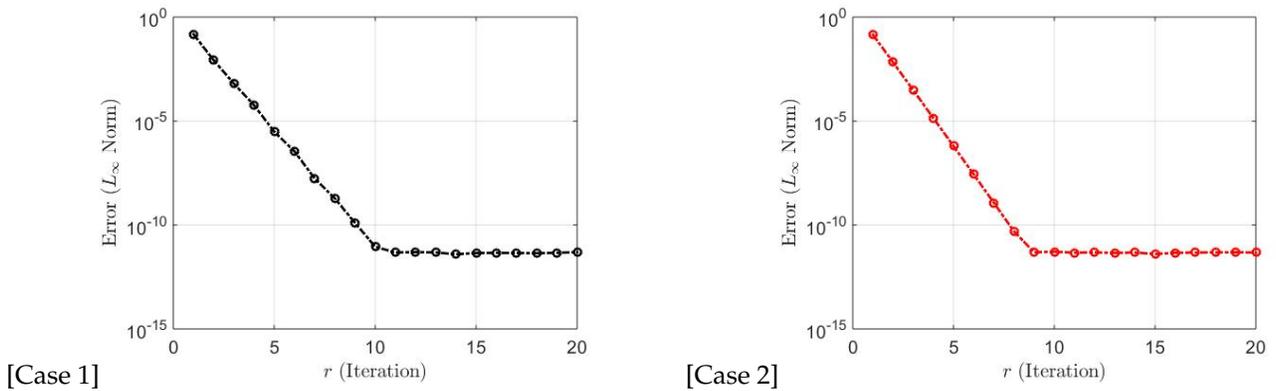
Fig. 8. Convergence graphs of the Burgers-Huxley equation.
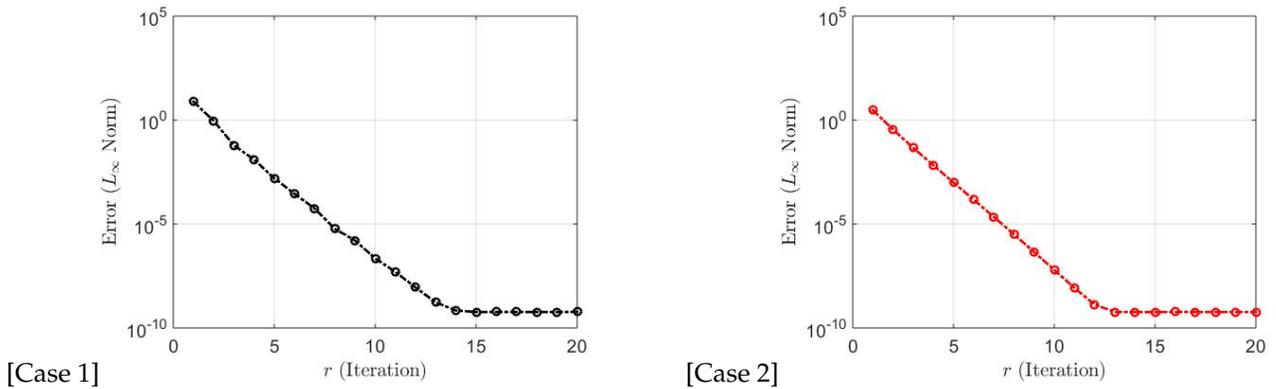
Fig. 9. Convergence graphs of Example 4.1 .

In Figures 7, 8, 9, convergence analysis graphs for the problems are presented. The figures present a variation of the error norm at a fixed value of time $(\xi = 0.5)$. As seen from the graphs, the BSLPM is convergent with less than 15 iterations, and the same slopes for all the graphs confirm that the rate of convergence of the method is the same. The method converges to an error of up to 10^{-14}, which is satisfactory and beyond the point where full convergence is reached, the error norm levels off and does not improve with an increase in the number of iterations. This plateau level gives a good estimate of the maximum error that can be achieved when using the method with a certain number of collocation points.

It is worth remarking that the accuracy of the method depends on the number of collocation points in both the ζ and ξ directions.

5. CONCLUSION

In this work, the bivariate spectral linear partition method, BSLPM, for solving nonlinear evolution differential equations. We discussed relevant literature that has motivated this work's development and presented previously used numerical methods, their advantages, and drawbacks, from which this work was developed. We then developed the proposed method in detail, for an nth order nonlinear parabolic equation. The method was then applied to selected evolution nonlinear equations and the numerical simulations and results obtained were presented and discussed in detail. The method was found to be accurate, with its solutions being compared with known exact solutions, and found to be in excellent agreement. Furthermore, the method was able to converge with a few grid points and to an error of up to 10^{-14}. These results were found to be comparable with the results of methods like BSQLM used on similar problems.

The success of this study suggests that the method can be used to analyze nonlinear problems existing in fluid mechanics and other fields. The method can also be used as a validating instrument for results generated using other numerical methods. Lastly, the bivariate spectral linear partition approach presented in this study contributes to the growing body of literature on numerical methods for solving complex nonlinear parabolic partial differential problems.

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