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Output Consensus Robustness and Performance of First-Order Agents †

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Abstract

In this paper, we study consensus robustness and performance problems for continuous-time multi-agent systems. We consider first-order unstable agents interconnected by an undirected graph, coordinated by a delayed output feedback protocol. Our objectives are twofold. First, we seek to determine the largest range of delay permissible so that the agents may achieve robustly consensus despite variation of the delay length, herein referred to as the delay consensus margin. Second, we attempt to determine consensus error performance quantified under an $\mathcal{H}_2$ norm criterion, which measures the disruptive effect of random nodal noises on consensus. For both problems, we obtain analytical solutions. The explicit expressions provide conceptual insights and exhibit how the agents' unstable pole, nonminimum phase zero, as well as the network topology may limit fundamentally the consensus robustness and performance.

KEYWORDS:

Time-delay systems, multi-agent systems, delay consensus robustness, delay consensus margin, consensus performance.

1 | INTRODUCTION

Multi-agent systems (MASs) are comprised of spatially distributed agents that communicate via a communication network to enable execution of joint, coordinated tasks. A central problem in the analysis and design of MASs is that of achieving consensus: The agents are to reach asymptotically a common state. Pertaining to this problem, agents of diverse dynamics have been investigated, including, e.g., single-integrators [1,2,3], double-integrators [4,5], Euler-Lagrangian systems [6] and more generally, linear and nonlinear agents [7,8]. The complexity of feedback protocols that enable the attainment of consensus range from simple static state feedback to dynamic, nonlinear, and adaptive output feedback. Applications of multi-agent consensus control are also widespread, in, e.g., formation flight [9], sensor networks [10], distributed computation, and biological systems, to name just a few.

Robustness and performance issues of MASs began to receive attention in the recent years. For example, system performance of MASs concerning leader-follower tracking was extensively investigated in , and performance on regulating energy consumption was explored in [11]. Also under heavy scrutiny are generic notions of consensus error performance; see, e.g., [12,13,14]. In these studies, it has been customary to incorporate transient response and examine the power of the agents subject to stochastic disturbances, which serves to quantify the dispersion of the noises propagated through the network. In particular, to focus on the fundamental characteristics of the underlying network, it has been common to study simple, low-order agents. Specifically, integrator agents were studied in [13,14] for networks described by an undirected graph, and in [12] for those described by a directed

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graph. Extensions to higher-order agents (e.g., double integrator agents) and those containing more problematic dynamic behaviors (e.g., unstable agents) were pursued in, e.g.,[15,16]. The studies have led to characterizations and interpretations of network metrics such as centrality, coherence, and network risk; see, e.g.,[15,17,18,19].

Of equal importance, robustness of MASs has also be extensively studied, often in the form of agent heterogeneity[20,21,22,23,24] or network transmission uncertainty[25,26,27]. Particularly relevant to the present work is a thread devoted to the study of consensus robustness with respect to uncertain inter-agent communication delays. Consensus robustness in this spirit has been investigated under the notion of *delay consensus margin (DCM)*[28,29], which constitutes the largest range of permissible delay values so that consensus may be achieved robustly within that range. Notably, for integrator and first-order agents with a static consensus protocol, analytical expressions of the DCM were obtained in[30] and[31], respectively. More general consensus feedback such as proportional-derivative and PID protocols were considered in[28,29], which result in readily computable solutions of the DCM, for first- and second-order agents. More generally, for heterogeneous delays, bounds on the DCM were derived in[32], where a frequency-sweeping method was proposed to estimate the delay range for consensus robustness. Time-varying delays were investigated, in, e.g.,[33,34].

In this paper we continue the aforementioned study on consensus error performance. We consider first-order unstable agents interconnected by an undirected graph subject to an unknown, uncertain constant delay, and we seek to determine the mean-square average consensus error under the disruptive influence of stochastic noises. The problem, being so formulated, provides a power measure for the error response, for which the exogenous stochastic signals can be interpreted as measurement noises at the nodes of the network, and the performance measure serves as one of resistance of the MAS in countering the nodal noises. For this purpose, we also attempt, as a precursor, to determine the DCM of the MAS. Our purpose is to investigate how agent dynamics and network topology may adversarily affect the DCM and the consensus performance. In a significant distinction from the previous work, we consider output feedback protocols. This allows us to examine unique issues of MASs unseen previously, of which the effect of an agent's nonminimum phase zero on consensus robustness and error performance is one of keen interest. On the other hand, to impose an output feedback protocol renders the problem more challenging. Indeed, as a direct consequence, the use of output feedback results in a *neutral* delay system, which is known to exhibit more intricate dynamic behaviors and system characteristics, and consequently pose significant complications in tackling the DCM and consensus performance problems.

The remainder of this paper is organized as follows. In section 2, we introduce the preliminary background required for our development, which consists of elementary graph theory and computation of the $\mathcal{H}_2$ norm of linear time-invariant (LTI) delay systems. The DCM and the consensus error performance problem are formulated in Section 3, where the exogenous stochastic signals are assumed to be uncorrelated white noises. Determining the consensus error performance is then seen to translate into the computation of the $\mathcal{H}_2$ norm of a delay MAS, and accordingly, the minimal consensus error under the power criterion amounts to solving an optimal $\mathcal{H}_2$ control problem. This leads to our main results, presented in Section 4. Analytical expressions are obtained for both the DCM and the optimal consensus error performance, where the former provides the fundamental limit of the delay range allowable for consensus attainment, and the latter furnishes the minimal $\mathcal{H}_2$ consensus error achievable, both of which shed useful light into limitations imposed by the agent dynamics and network topology on the robustness and performance of first-order MASs. Illustrative examples are given in Section 5, and the paper concludes in Section 6.

The notation used throughout this paper is as follows. For any number z, any vector u, and any matrix A, we denote by z^T, u^T, and A^T their transposes, respectively. For any matrix $A = [a_{ij}]$, we denote its Frobenius norm by $\|A\|_F$. For a square matrix A, we denote its trace by $\mathrm{Tr}(A)$, and its pseudo-inverse by A^+. We write $A \geq 0$ if A is nonnegative definite. The Hölder ℓ_2 norm of a vector u is denoted by $\|u\|$. For a vector signal $u(t)$ defined on $[0,\ \infty)$, we denote its $\mathcal{L}_2$ norm by $\|u\|_2$, where

$$\|u\|_2^2 = \int_0^\infty \|u(t)\|^2 \mathrm{d}t.$$

The $\mathcal{L}_2$ norm of a matrix function $F(j\omega)$ defined on the imaginary axis is defined by

$$\|F\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|F(j\omega)\|_F^2 \mathrm{d}\omega.$$

For a stable transfer function matrix $F(s)$, the $\mathcal{L}_2$ norm coincides with its $\mathcal{H}_2$ norm. Note that while we use the same notation $\|\cdot\|_2$ for $\mathcal{L}_2$ and $\mathcal{H}_2$ norms, the distinction will be self-evident from the context. Finally, we denote the expectation operator by $\mathbb{E}\{\cdot\}$.

2 | PRELIMINARIES AND PROBLEM FORMULATION

2.1 | Elementary Graph Theory

We begin by introducing some basics of graph theory[35]. It is routine to associate a communication network with a graph represented by $\mathcal{G} = (\mathcal{V},\ \mathcal{E},\ \mathcal{A})$. Here $\mathcal{V} = \{v_1, \cdots, v_N\}$ is the node set with each node v_i representing an agent, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is an *edge* set of paired nodes. If an edge $(v_i,\ v_j) \in \mathcal{E}$, the jth node can obtain information from the ith node. This access to information need not be reciprocal, depending on whether the graph is *directed* or *undirected*. A graph is said to be *undirected* if for all $v_i,\ v_j \in \mathcal{V}$, $(v_j, v_i) \in \mathcal{E}$ implies that $(v_i, v_j) \in \mathcal{E}$; otherwise, the graph is said to be *directed*. Throughout this paper, we consider undirected graphs. A *path* from node v_1 to node v_k is a sequence of nodes $v_1, \cdots, v_k$, such that for each i, $1 \le i \le k-1$, (v_i, v_{i+1}) is an edge. A graph is said to be *connected* if there exists a path from any node to any other node, and *complete* if every pair of distinct nodes is connected by an edge. We assume that the graph under consideration is connected. A graph $\mathcal{G}$ can be represented by its adjacency matrix $\mathcal{A} = [a_{ij}]$, whose element a_{ij} corresponds to the adjacent nodes $(v_i,\ v_j)$. Define the in-degree matrix $\mathcal{D} \triangleq \mathrm{diag}(\sum_j a_{1j},\ \cdots,\ \sum_j a_{Nj})$. Then, the graph can also be equivalently described by its Laplacian matrix $\mathcal{L} = \mathcal{D} - \mathcal{A}$. It is well-known that for an undirected graph $\mathcal{G}$, the Laplacian matrix $\mathcal{L}$ is symmetric and nonnegative definite. Under this circumstance, $\mathcal{L}$ admits a unitary decomposition $\mathcal{L} = \mathcal{W}\Lambda\mathcal{W}^T$, where Λ is a diagonal matrix, i.e., $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ with $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$. The matrix $\mathcal{W} = [w_1\ w_2\ \cdots\ w_N]$ is a unitary matrix with $w_1 = (1/\sqrt{N})\mathbf{1}_N$, where $\mathbf{1}_N = [1\ 1\ \cdots\ 1]^T$. It follows that $w_i^T \mathbf{1}_N = 0$ for $i = 2, \cdots, N$.

2.2 | $\mathcal{H}_2$ norm of Delay System

Consider the linear time-invariant neutral delay system

$$\begin{aligned} B_0\dot{x}(t) + B_1\dot{x}(t-\tau) &= A_0 x(t) + A_1 x(t-\tau) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t)$ represents the state, $u(t)$ the input, and $y(t)$ the output of the system, respectively. B_0, B_1, A_0, A_1, B and C are real matrix of appropriate dimensions. The parameter $\tau > 0$ is a constant but unknown delay. Let the system be stable and define its transfer function

$$H(s) = C\left(\left(B_0 + B_1 e^{-\tau s}\right)s - \left(A_0 + A_1 e^{-\tau s}\right)\right)^{-1} B.$$

Define the output and input *delay Lyapunov matrix*[36] by

$$U(t) = \int_0^\infty \Phi^T(s) C^T C \Phi(s+t) \mathrm{d}t, \tag{2}$$

$$V(t) = \int_0^\infty \Phi^T(s) B B^T \Phi(s+t) \mathrm{d}t, \tag{3}$$

where $\Phi(t)$ is the fundamental solution matrix to the system (1). The following result, adapted from[37], characterizes the system's $\mathcal{H}_2$ norm in terms of solutions to delayed Lyapunov equations.

Lemma 1. Suppose that the system (1) is stable. Then the $\mathcal{H}_2$ norm of the transfer function matrix $H(s)$ is given by

$$\|H\|_2^2 = \mathrm{Tr}\,(B^T U(0) B) = \mathrm{Tr}\,(C V(0) C^T), \tag{4}$$

where $U(t)$ is the unique solution to the delayed Lyapunov equation

$$\begin{aligned} \dot{U}(t)B_0 + \dot{U}(t-\tau)B_1 &= U(t)A_0 + U(t-\tau)A_1, \\ U(-t) &= U^T(t), \\ -C^T C &= \sum_{i=0}^{1}\sum_{i=0}^{1}\left(B_i^T U(\tau_i - \tau_j)A_j + A_j^T U^T(\tau_i - \tau_j)B_i\right), \end{aligned} \tag{5}$$

with $\tau_0 = 0$, and $V(t)$ is the unique solution to the delayed Lyapunov equation

$$\begin{aligned} \dot{V}(t) + \dot{V}(t-\tau)B_1^T &= V(t)A_0^T + V(t-\tau)A_1^T, \\ V(-t) &= V^T(t), \\ -BB^T &= \sum_{i=0}^{1}\sum_{i=0}^{1}\left(B_i V(\tau_i - \tau_j)A_j^T + A_j V^T(\tau_i - \tau_j)B_i^T\right). \end{aligned} \tag{6}$$

2.3 | The Consensus Problem

We consider continuous-time agents with dynamics described by the transfer function

$$p(s) = \frac{\alpha s - \beta}{s - p}, \tag{7}$$

where $\alpha \geq 0$, $p \geq 0$, and β is a real number that can be both nonnegative and negative. The agents can also be described by the state-space equation

$$\begin{aligned} \dot{x}_i(t) &= p x_i(t) + u_i(t), \\ y_i(t) &= (\alpha p - \beta)x_i(t) + \alpha u_i(t), \end{aligned} \tag{8}$$

for $i = 1, \cdots, N$. With this formulation, the agents are unstable and may or may not be nonminimum phase, depending on whether $\beta > 0$ or $\beta \leq 0$. Note that with different combinations of α, β, and p, the formulation reduces to some of the agent models frequently studied previously; for example, for $\beta = 0$, the consensus problem was studied in[28,29], and if further $p = 0$, the agents become pure integrators.

We consider the output feedback control protocol

$$u_i(t) = -k\sum_{j=1}^{N} a_{ij}\left(y_i(t-\tau) - y_j(t-\tau)\right) + v_i(t). \tag{9}$$

In this feedback law, the agent's input undergoes a constant delay $\tau \geq 0$. The exogenous signal $v_i(t)$ can be interpreted as a disturbance signal or a measurement noise. Define the average state of the MAS by

$$\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t),$$

and the error responses by

$$e_i(t) = x_i(t) - \bar{x}(t), \quad i = 1, \cdots, N,$$

With the output feedback protocol (9), we say that the MAS (8) achieves consensus over the graph $\mathcal{G}$, represented by the adjacency matrix $\mathcal{A}$, if

$$\lim_{t\to\infty} \left|e_i(t)\right| = 0, \quad \forall i = 1, \cdots, N. \tag{10}$$

In the sequel, one of the primary objectives is to investigate how the delay in the feedback protocol (9) may affect consensus, and determine accordingly the largest range of this delay so that consensus can be achieved robustly. Another objective is to quantify the consensus error responses under the effect of the noise signals $v_i(t)$. For these purposes, we shall first derive the closed-loop error response equation. Denote

$$\begin{aligned} x(t) &= [x_1(t), x_2(t), \cdots, x_N(t)]^T, \\ y(t) &= [y_1(t), y_2(t), \cdots, y_N(t)]^T, \\ u(t) &= [u_1(t), u_2(t), \cdots, u_N(t)]^T, \\ v(t) &= [v_1(t), v_2(t), \cdots, v_N(t)]^T, \end{aligned}$$

and correspondingly their Laplace transforms by $X(s)$, $Y(s)$, $U(s)$, and $V(s)$, respectively. Then it follows from (8) and (9) that

$$\begin{aligned} Y(s) &= (\alpha p - \beta)X(s) + \alpha U(s), \\ U(s) &= -k\mathcal{L}Y(s)e^{-\tau s} + V(s). \end{aligned}$$

Solving for $U(s)$ yields

$$U(s) = -\left[I + \alpha k\mathcal{L}e^{-\tau s}\right]^{-1} k(\alpha p - \beta)\mathcal{L}e^{-\tau s}X(s) + \left[I + \alpha k\mathcal{L}e^{-\tau s}\right]^{-1} V(s).$$

This, along with (8), gives rise to

$$\dot{x}(t) + \alpha k \mathcal{L} \dot{x}(t-\tau) = p x(t) + \beta k \mathcal{L} x(t-\tau) + v(t). \tag{11}$$

Note that whenever $\alpha \neq 0$, that is, when the agent dynamics contain a finite zero, then the MAS as described by (11) results in a neutral delay system, due to the use of output feedback protocol.

3 | DELAY CONSENSUS MARGIN

With the agents given by (8) and the consensus feedback protocol given by (9), we define the delay consensus margin (DCM) by

$$\bar{\tau} = \sup\{\mu \geq 0 : \text{There exists a } k \text{ such that consensus of MAS (8) is achieved under the protocol (9) } \forall \tau \in [0,\ \mu)\}.$$

Clearly, the DCM $\bar{\tau}$ defines the largest range of delay within which the consensus can be achieved by the proportional output feedback protocol, for all $\tau \in [0,\ \bar{\tau})$. Denote similarly $e(t) = [e_1(t), e_2(t), \cdots, e_N(t)]^T$, and consider $\tilde{e}(t) = \mathcal{W}^T e(t)$. Evidently, $\lim_{t\to\infty} |e(t)| = 0$ if and only if $\lim_{t\to\infty} |\tilde{e}(t)| = 0$. Hence, for consensus, we may consider the error signal $\tilde{e}(t)$. Furthermore, with no loss of generality, we may assume that $v(t) = 0$. From (11), it then follows that

$$\dot{\tilde{e}}(t) + \alpha k \Lambda \dot{\tilde{e}}(t-\tau) = p\tilde{e}(t) + \beta k \Lambda \tilde{e}(t-\tau).$$

Since $\tilde{e}_1(t) = \frac{1}{\sqrt{N}} \mathbf{1}_N^T e(t) = \frac{1}{\sqrt{N}} \mathbf{1}_N^T \left(I - \frac{1}{N}\mathbf{1}_N \mathbf{1}_N^T\right) x(t) = 0$, it suffices to consider

$$\dot{\tilde{e}}_i(t) + \alpha k \lambda_i \dot{\tilde{e}}_i(t-\tau) = p\tilde{e}_i(t) + \beta k \lambda_i \tilde{e}_i(t-\tau), \quad i = 2,\ \cdots,\ N. \tag{12}$$

In other words, the consensus of MAS (8) is achieved under the protocol (9) if and only if the delay systems in (12) are all stable. For the latter to be true, it is necessary for the discrete parts of the neutral system, that is, the systems

$$\tilde{e}_i(t) + \alpha k \lambda_i \tilde{e}_i(t-\tau) = 0, \quad i = 2,\ \cdots,\ N,$$

to be stable[38], which requires that

$$\alpha\, |k|\, \lambda_i < 1, \quad i = 2,\ \cdots,\ N. \tag{13}$$

Moreover, for the MAS (8) to achieve consensus over $[0,\ \bar{\tau})$ for some $\bar{\tau}$, the systems in (12) must also be stable at $\tau = 0$. Under the condition (13), this means that

$$\beta k \lambda_i < -p, \quad i = 2,\ \cdots,\ N. \tag{14}$$

The conditions (13) and (14) together suggest that $p/(|\beta|\lambda_2) < |k| < 1/(\alpha\lambda_N)$. Define

$$\Omega = \left\{ k : \frac{p}{|\beta|\lambda_2} < |k| < \frac{1}{\alpha\lambda_N} \right\}.$$

It follows that for the MAS (8) to achieve consensus, it is necessary that $k \in \Omega$. Note that for Ω not to be empty, necessarily,

$$\frac{\lambda_N}{\lambda_2} < \frac{|\beta|}{\alpha p}, \tag{15}$$

thus indicating that the presence of the nonminimum phase zero imposes a stringent constraint on the eigen-ratio λ_N/λ_2, widely known as a measure of network connectivity. We shall assume that this condition is met throughout the paper.

The following result provides an analytical expression of the DCM.

Theorem 1. Let $\beta > 0$ and suppose that the condition (15) hold. Then for a connected undirected graph $\mathcal{G}$, the DCM of MAS (12) under the protocol (9) is

$$\bar{\tau} = \frac{\arctan\left(\gamma \frac{1 - \frac{\alpha^2 p^2 \lambda_N^2}{\beta^2 \lambda_2^2}}{1 + \frac{\alpha p \lambda_N^2}{\beta \lambda_2^2}}\right)}{p\gamma}, \tag{16}$$

where

$$\gamma = \sqrt{\frac{\frac{\lambda_N^2}{\lambda_2^2} - 1}{1 - \frac{\alpha^2 p^2 \lambda_N^2}{\beta^2 \lambda_2^2}}}. \tag{17}$$

It is clear from Theorem 1 that for first-order multi-agent (8), DCM is achieved at the boundary of its allowable range Ω. The explicit expression of DCM quantifies the effect of the agents' unstable poles, nonminimum phase zeros, and the eigenvalues ratio of the Laplacian matrix on consensus of delay system. Theorem 1 also leads to several useful observations.

Remark 1. For first-order agents with relative degree one, i.e., when $\alpha = 0$, the DCM becomes

$$\bar{\tau} = \frac{\arctan\left(\sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}\right)}{p\sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}}. \tag{18}$$

This result was previously obtained in[31,29]. In comparison, it is easy to see that in the presence of the nonminimum phase zero β/α, $\beta > 0$, the DCM is reduced. This can be seen by noting that $\bar{\tau}$ can be written alternatively as

$$\bar{\tau} = \frac{\arctan\left(\sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}\,\frac{\sqrt{1-\frac{\alpha^2 p^2 \lambda_N^2}{\beta^2 \lambda_2^2}}}{1+\frac{\alpha p \lambda_N^2}{\beta \lambda_2^2}}\right)}{p\sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}\sqrt{1 - \frac{\alpha^2 p^2 \lambda_N^2}{\beta^2 \lambda_2^2}}}. \tag{19}$$

Evidently,

$$\frac{\sqrt{1 - \frac{\alpha^2 p^2 \lambda_N^2}{\beta^2 \lambda_2^2}}}{1 + \frac{\alpha p \lambda_N^2}{\beta \lambda_2^2}} \leq 1.$$

Hence,

$$\bar{\tau} \leq \frac{\arctan\left(\sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}\right)}{p\sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}\sqrt{1 - \frac{\alpha^2 p^2 \lambda_N^2}{\beta^2 \lambda_2^2}}} \leq \frac{\arctan\left(\sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}\right)}{p\sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}}.$$

Remark 2. Consider more specifically the case $\alpha = 1$ and $\beta = z > 0$. Under this circumstance, the expression (19) reduces to

$$\bar{\tau} = \frac{\arctan\left(\sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}\,\frac{\sqrt{1-\left(\frac{p}{z}\right)^2 \frac{\lambda_N^2}{\lambda_2^2}}}{1+\left(\frac{p}{z}\right)\frac{\lambda_N^2}{\lambda_2^2}}\right)}{p\sqrt{\frac{\lambda_N^2}{\lambda_2^2} - 1}\sqrt{1 - \left(\frac{p}{z}\right)^2 \frac{\lambda_N^2}{\lambda_2^2}}}. \tag{20}$$

It is clear that $\bar{\tau}$ is monotonically decreasing with respect to p/z. Hence, as expected intuitively, when the unstable pole p and the nonminimum phase zero z become closer, a smaller DCM results. In the limit when $p/z \to 0$, the DCM approaches that in the absence of nonminimum phase zero, i.e., (18), while as $p/z \to \lambda_2/\lambda_N$, $\bar{\tau} \to 0$. Note also that for a complete graph, that is, if $\lambda_2 = \cdots = \lambda_N$, then by taking the limit with $\lambda_N/\lambda_2 \to 1$, the DCM is found to be

$$\bar{\tau} = \frac{1}{p} - \frac{1}{z},$$

which coincides with the delay margin of a first-order system with one unstable pole and one nonminimum phase zero[39,40].

The rest of this section is devoted to the proof of Theorem 1. Before proceeding, we first need the following preliminary lemma.

Lemma 2. i) For any $\zeta \geq 0$ and $\eta \geq 0$,

$$\arctan\zeta - \arctan\eta = \arctan\frac{\zeta - \eta}{1 + \zeta\eta}. \tag{21}$$

ii) For any $a > 0$ and $b > 0$ such that $a < b$, the function

$$f(\omega) = \frac{\arctan(b\omega) - \arctan(a\omega)}{\omega} \tag{22}$$

is monotonically decreasing with respect to $\omega \geq 0$.

Proof. The statement i) is a well-known property of the arctangent function, which can be found in, e.g.,[41]. To prove ii), we evaluate the derivative of $f(\omega)$ with respect to ω, which is given by

$$f'(\omega) = \frac{\frac{b\omega}{1+(b\omega)^2} - \arctan(b\omega) - \frac{a\omega}{1+(a\omega)^2} + \arctan(a\omega)}{\omega^2}. \tag{23}$$

Denote

$$g(x) = \frac{x}{1+x^2} - \arctan x.$$

It is easy to see that $g(x)$ is monotonically decreasing with respect to $x \geq 0$. Indeed, this can be readily verified by taking the derivative of $g(x)$, yielding

$$g'(x) = \frac{1-x^2}{(1+x^2)^2} - \frac{1}{1+x^2} = -\frac{2x^2}{(1+x^2)^2} \leq 0.$$

As a result, for $a < b$, $g(b\omega) \leq g(a\omega)$, and hence $f'(\omega) = (g(b\omega) - g(a\omega))/\omega^2 \leq 0$; that is, $f(\omega)$ is monotonically decreasing with respect to $\omega \geq 0$. ■

We are now ready to present the proof for Theorem 1.

Proof of Theorem 1. The characteristic quasipolynomials of the systems in (12) are readily found as

$$q_i(s,\ e^{-\tau s}) = s + \alpha k \lambda_i s e^{-\tau s} - p - \beta k \lambda_i e^{-\tau s}, \quad i = 2, \cdots, N.$$

Under the condition (13), a necessary and sufficient condition[38] for the systems to be stable is that the quasipolynomials $q_i(s,\ e^{-\tau s})$ are stable for all $i = 2, \cdots, N$, that is, $q_i(s,\ e^{-\tau s}) \neq 0$ for all $s \in \overline{\mathbb{C}}_+$. For any $k \in \Omega$, $q_i(s,\ e^{-\tau s})$ is stable at $\tau = 0$. Hence by continuity, it becomes unstable for some $\tau > 0$ whenever $q_i(j\omega,\ e^{-j\tau\omega}) = 0$. Denote

$$L_i(s) = \frac{(\alpha s - \beta)k\lambda_i}{s - p},$$

Then equivalently, $q_i(j\omega,\ e^{-j\tau\omega}) = 0$ if and only if

$$1 + L_i(j\omega)e^{-j\tau\omega} = 0. \tag{24}$$

Find the frequency $\omega_i(k)$ such that $|L_i(j\omega_i(k))| = 1$, which is given by the solution

$$\omega_i^2(k) = \frac{\beta^2 k^2 \lambda_i^2 - p^2}{1 - \alpha^2 k^2 \lambda_i^2}, \tag{25}$$

for any $k \in \Omega$. At $\omega = \omega_i(k)$, we obtain

$$\sphericalangle L_i(j\omega_i(k)) = -\pi + \arctan\frac{\omega_i(k)}{p} - \arctan\frac{\alpha\omega_i(k)}{\beta}.$$

Select $\tau_i(k)$ such that

$$\omega_i(k)\tau_i(k) = \arctan\frac{\omega_i(k)}{p} - \arctan\frac{\alpha\omega_i(k)}{\beta}.$$

It follows that $1 + L_i(j\omega_i(k))e^{-j\tau_i(k)\omega_i(k)} = 0$, and that for any $\omega \geq 0$, $1 + L_i(j\omega)e^{-j\tau\omega} \neq 0$ for all $\tau \in [0,\ \tau_i(k))$. Thus, the largest range of delay such that $q_i(s,\ e^{-\tau s})$ are stable for all $i = 2, \cdots, N$ is $\min_{2\leq i\leq N} \tau_i(k)$, and in turn the DCM is found as

$$\bar{\tau} = \sup_{k\in\Omega} \min_{2\leq i\leq N} \tau_i(k) = \sup_{k\in\Omega} \min_{2\leq i\leq N} \frac{\arctan\frac{\omega_i(k)}{p} - \arctan\frac{\alpha\omega_i(k)}{\beta}}{\omega_i(k)}.$$

According to (15), $\alpha/\beta \leq 1/p$. Hence, in light of Lemma 2 ii), $\tau_i(k)$ is a monotonically decreasing function of $\omega_i(k)$. Furthermore, it is easy to verify that $\omega_i(k)$ increase monotonically with λ_i. As a result,

$$\min_{2\leq i\leq N} \tau_i(k) = \frac{\arctan\frac{\omega_N(k)}{p} - \arctan\frac{\alpha\omega_N(k)}{\beta}}{\omega_N(k)} = \tau_N(k),$$

and

$$\bar{\tau} = \sup_{k\in\Omega} \frac{\arctan \frac{\omega_N(k)}{p} - \arctan \frac{\alpha\omega_N(k)}{\beta}}{\omega_N(k)}$$
$$= \tau_N\left(\frac{p}{\beta\lambda_2}\right),$$

where the supremum is achieved at $k = p/(\beta\lambda_2)$, again by the monotonicity of $\tau_i(k)$ with respect to k. The proof can then be completed by noting that

$$\omega_N\left(\frac{p}{\beta\lambda_2}\right) = p\sqrt{\frac{\frac{\lambda_N^2}{\lambda_2^2} - 1}{1 - \frac{\alpha^2p^2\lambda_N^2}{\beta^2\lambda_2^2}}},$$

and by incoking Lemma 2 i). ■

In summary, Theorem 1 indicates that with output consensus feedback, the nonminimum phase zero of the agents will invariably constrain the DCM achievable, which corresponds to $\beta > 0$. Analogously, by mimicking the proof of Theorem 1, one may also obtain the expression of DCM in the case $\beta < 0$, that is, in the presence of a minimum phase zero $|\beta|/\alpha$, found as

$$\bar{\tau} = \frac{\arctan\left(\gamma \frac{1-\frac{\alpha^2p^2\lambda_N^2}{\beta^2\lambda_2^2}}{1-\frac{\alpha p\lambda_N^2}{|\beta|\lambda_2^2}}\right)}{p\gamma}$$

if $\alpha p\lambda_N^2/(|\beta|\lambda_2^2) < 1$, and otherwise

$$\bar{\tau} = \frac{\pi + \arctan\left(\gamma \frac{1-\frac{\alpha^2p^2\lambda_N^2}{\beta^2\lambda_2^2}}{1-\frac{\alpha p\lambda_N^2}{|\beta|\lambda_2^2}}\right)}{p\gamma},$$

where γ is given in (17). In either case, it is easy to see that the DCM is greater than that in (16).

4 | CONSENSUS ERROR PERFORMANCE

Having determined the delay consensus margin, in this section we attempt to quantify the minimal consensus error under criterion compatible with external noises. The problem under consideration concerns the disruption effect of the noises on consensus. We consider specifically random noises characterized by the following assumptions:

Assumption 1. Each component $\xi_i(t)$ of $\xi(t) = [\xi_1(t), \cdots, \xi_N(t)]^T$ is zero mean, $\mathbb{E}\{\xi_i(t)\} = 0$, and

$$\mathbb{E}\{|\xi_i(s)|^2\} \le \sigma_i^2\delta(t-s),$$

where $\delta(t)$ is the Dirac function, and σ_i^2 is a given bound on the variance of $\xi_i(t)$.

Assumption 2. $\{\xi_i(t)\}$ and $\xi_j(t)$ are uncorrelated processes for $i \neq j$, i.e.,

$$\mathbb{E}\{\xi_i(t)\xi_j(s)\} = 0, \quad \forall t,\ s \text{ and } i \neq j.$$

With these assumptions, $\xi(t)$ may be considered a continuous-time white noise, resembling its discrete-time counterpart. We denote the class of all signals satisfying Assumption 1 and Assumption 2 by the set Ξ.

For the given Laplacian matrix $\mathcal{L}$, decompose the unitary matrix $\mathcal{W}$ as

$$\mathcal{W} = \left[\frac{1}{\sqrt{N}}\mathbf{1}_N \ Q\right]. \tag{26}$$

Let $v(t) = Q\xi(t)$, with $\xi(t) \in \Xi$. We may rewrite (11) as

$$\dot{x}(t) + \alpha kL\dot{x}(t-\tau) = px(t) + \beta kLx(t-\tau) + Q\xi(t), \tag{27}$$

For any $k \in \Omega$ and $\tau \in [0,\ \bar{\tau})$, we are interested in finding

$$E = \sup_{\xi(t)\in\Xi} \mathbb{E}\{\|e(t)\|_2^2\}.$$

This amounts to determining the variance of the consensus error response under the protocol (9), due to the disruptive effect of all possible random noises in the class Ξ, propagated through the network topology represented by Q. The measure thus characterizes the mean-square performance achievable by delayed output proportional feedback protocol.

Theorem 2. Let $\beta > 0$ and $\mathcal{G}$ be a connected undirected graph. Then under Assumption 1 and Assumption 2, the mean-square consensus error performance, for any $k \in \Omega$ and $\tau \in [0,\ \bar{\tau})$, is given by

$$E = \frac{1}{2(\beta+\alpha p)|k|} \sum_{i=2}^{N} \frac{\cos(\Gamma_i\tau+\theta_i)\sigma_i^2}{\lambda_i\cos\theta_i(1-\sin(\Gamma_i\tau+\theta_i))}, \tag{28}$$

where

$$\begin{aligned}\Gamma_i &= \sqrt{\frac{\beta^2k^2\lambda_i^2-p^2}{1-\alpha^2k^2\lambda_i^2}},\\ \theta_i &= \cos^{-1}\frac{\Gamma_i(1-\alpha^2k^2\lambda_i^2)}{|k|\lambda_i(\beta+\alpha p)}.\end{aligned}$$

Proof. We first note that $Q \in \mathbb{R}^{N\times(N-1)}$ satisfies the following properties:

$$\begin{aligned}&Q^*\mathbf{1}_N = 0,\\ &Q^*Q = I_{N-1},\\ &QQ^* = I_N - \tfrac{1}{N}\mathbf{1}_N\mathbf{1}_N^T.\end{aligned} \tag{29}$$

As a result, we have $e(t) = QQ^*x(t)$, and $\|e(t)\|_2 = \|Q^*x(t)\|_2$. Let $z(t) = Q^*x(t)$. Then the signal $z(t)$, together with the equation (27), forms a state-space description with input $\xi(t)$ and output $z(t)$, whose transfer function matrix is given by

$$\begin{aligned}H(s) &= Q^*(sI - pI + \alpha kLse^{-\tau s} - \beta kLe^{-\tau s})^{-1}Q\\ &= \left((s-p)I + (\alpha s-\beta)k\bar{\Lambda}e^{-\tau s}\right)^{-1},\end{aligned}$$

where $\bar{\Lambda} = \mathrm{diag}(\lambda_2,\cdots,\lambda_N)$. It thus follows that $\|e(t)\|_2 = \|Q^*x(t)\|_2 = \|z(t)\|_2$. Under Assumption 1 and Assumption 2, this leads to

$$\mathbb{E}\{\|z(t)\|_2^2\} = \sum_{i=2}^{N}\|G_i\|_2^2\mathbb{E}\{\|\xi(t)\|_2^2\},$$

where $G_i(s) = G(s,\ \lambda)$ with $\lambda = \lambda_i$, and

$$G(s,\ \lambda) = \frac{1}{s-p+(\alpha s-\beta)k\lambda e^{-\tau s}}.$$

Evidently,

$$\sup_{\xi(t)\in\Xi}\mathbb{E}\{\|z(t)\|_2^2\} = \sum_{i=2}^{N}\sigma_i^2\|G_i\|_2^2.$$

We proceed to compute $\|G_i\|_2$. For this purpose, we note that $G(s)$ admits a state-space realization given by (1), with $A_0 = p$, $A_1 = \beta k\lambda$, $B_0 = 1$, $B_1 = \alpha k\lambda$, $B = 1$, and $C = 1$. The delay Lyapunov matrix is given by

$$\phi(t) = \int_0^\infty \mu(r)\mu(r+t)\mathrm{d}r = \int_t^\infty \mu(r-t)\mu(r)\mathrm{d}r,$$

where

$$\dot{\mu}(t) + \alpha k\lambda\dot{\mu}(t-\tau) = p\mu(t) + \beta k\lambda\mu(t-\tau);$$

that is, $\mu(t)$ is the fundamental solution. According to Lemma 1, $\phi(t)$ satisfies the delay Lyapunov equation

$$\dot{\phi}(t) + \alpha k\lambda\dot{\phi}(t-\tau) = p\phi(t) + \beta k\lambda\phi(t-\tau).$$

Define $\psi(t) = \phi(\tau - t) = \phi(t-\tau)$. Then the equation can be written as

$$\dot{\phi}(t) + \alpha k\lambda\dot{\psi}(t) = p\phi(t) + \beta k\lambda\psi(t). \tag{30}$$

Furthermore,

$$\begin{aligned}\dot\psi(t) &= -\dot\phi(\tau-t) = \alpha k\lambda\dot\phi(-t) - pU(\tau-t) - \beta k\lambda\phi(-t)\\ &= -\alpha k\lambda\dot\phi(t) - p\psi(t) - \beta k\lambda\phi(t).\end{aligned} \tag{31}$$

The equations (30) and (31) can together be written as

$$\begin{bmatrix} 1 & \alpha k\lambda \\ \alpha k\lambda & 1 \end{bmatrix}\begin{bmatrix}\dot\phi(t)\\ \dot\psi(t)\end{bmatrix} = \begin{bmatrix} p & \beta k\lambda \\ -\beta k\lambda & -p \end{bmatrix}\begin{bmatrix}\phi(t)\\ \psi(t)\end{bmatrix},$$

or equivalently,

$$\begin{bmatrix}\dot\phi(t)\\ \dot\psi(t)\end{bmatrix} = \frac{1}{1-\alpha^2k^2\lambda^2}\begin{bmatrix} p+\alpha\beta k^2\lambda^2 & k\lambda(\beta+\alpha p) \\ -k\lambda(\beta+\alpha p) & -(p+\alpha\beta k^2\lambda^2) \end{bmatrix}\begin{bmatrix}\phi(t)\\ \psi(t)\end{bmatrix}.$$

Denote by $\Phi(s)$ and $\Psi(s)$ the Laplace transforms of $\phi(t)$ and $\psi(t)$. Then,

$$\begin{bmatrix}\Phi(s)\\ \Psi(s)\end{bmatrix} = \frac{1}{D(s)}\begin{bmatrix} s+\frac{p+\alpha\beta k^2\lambda^2}{1-\alpha^2k^2\lambda^2} & \frac{k\lambda(\beta+\alpha p)}{1-\alpha^2k^2\lambda^2} \\ -\frac{k\lambda(\beta+\alpha p)}{1-\alpha^2k^2\lambda^2} & s-\frac{p+\alpha\beta k^2\lambda^2}{1-\alpha^2k^2\lambda^2} \end{bmatrix}\begin{bmatrix}\phi(0)\\ \psi(0)\end{bmatrix}, \tag{32}$$

where

$$D(s) = s^2 + \frac{\beta^2k^2\lambda^2 - p^2}{1-\alpha^2k^2\lambda^2}.$$

For any $k \in \Omega$ and $\lambda = \lambda_i$, $i = 2, \cdots, N$, $\beta^2k^2\lambda^2 - p^2 > 0$ and $1-\alpha^2k^2\lambda^2 > 0$. Denote

$$\Gamma = \sqrt{\frac{\beta^2k^2\lambda^2 - p^2}{1-\alpha^2k^2\lambda^2}}\,.$$

Taking inverse Laplace transform, we find

$$\begin{bmatrix}\phi(\tau)\\ \psi(\tau)\end{bmatrix} = \begin{bmatrix} \cos(\Gamma\tau)+\frac{p+\alpha\beta k^2\lambda^2}{\Gamma(1-\alpha^2k^2\lambda^2)}\sin(\Gamma\tau) & \frac{k\lambda(\beta+\alpha p)}{\Gamma(1-\alpha^2k^2\lambda^2)}\sin(\Gamma\tau) \\ -\frac{k\lambda(\beta+\alpha p)}{\Gamma(1-\alpha^2k^2\lambda^2)}\sin(\Gamma\tau) & \cos(\Gamma\tau)-\frac{p+\alpha\beta k^2\lambda^2}{\Gamma(1-\alpha^2k^2\lambda^2)}\sin(\Gamma\tau) \end{bmatrix}\begin{bmatrix}\phi(0)\\ \psi(0)\end{bmatrix}.$$

Noting that $\psi(\tau) = \phi(0)$ and $\psi(0) = \phi(\tau)$, we obtain

$$\phi(0) = \frac{\cos(\Gamma\tau)-\frac{p+\alpha\beta k^2\lambda^2}{\Gamma(1-\alpha^2k^2\lambda^2)}\sin(\Gamma\tau)}{1+\frac{k\lambda(\beta+\alpha p)}{\Gamma(1-\alpha^2k^2\lambda^2)}\sin(\Gamma\tau)}\phi(\tau) = \frac{\Gamma(1-\alpha^2k^2\lambda^2)\cos(\Gamma\tau)-(p+\alpha\beta k^2\lambda^2)\sin(\Gamma\tau)}{\Gamma(1-\alpha^2k^2\lambda^2)+k\lambda(\beta+\alpha p)\sin(\Gamma\tau)}\phi(\tau). \tag{33}$$

By a direct calculation, we find that

$$\Gamma^2(1-\alpha^2k^2\lambda^2)^2 + (p+\alpha\beta k^2\lambda^2)^2 = k^2\lambda^2(\beta+\alpha p)^2.$$

Define

$$\theta = \cos^{-1}\frac{\Gamma(1-\alpha^2k^2\lambda^2)}{|k|\lambda(\beta+\alpha p)}.$$

Since $k < 0$ by (14), the equation (33) can be rewritten as

$$\phi(0) = \frac{\cos\theta\cos(\Gamma\tau)-\sin\theta\sin(\Gamma\tau)}{\cos\theta-\sin(\Gamma\tau)}\phi(\tau) = \frac{\cos(\Gamma\tau+\theta)}{\cos\theta-\sin(\Gamma\tau)}\phi(\tau). \tag{34}$$

Meanwhile, from (5), $\phi(\tau)$ and $\phi(0)$ are related by the equation

$$\phi(\tau) + \frac{p+\alpha\beta k^2\lambda^2}{(\beta+\alpha p)k\lambda}\phi(0) = -\frac{1}{2(\beta+\alpha p)k\lambda}.$$

or equivalently,

$$\phi(\tau) = \sin\theta\phi(0) + \frac{1}{2(\beta+\alpha p)|k|\lambda}. \tag{35}$$

Substituting (35) into (34), we then arrive at the solution

$$\phi(0) = \left(\frac{1}{2(\beta+\alpha p)|k|\lambda}\right)\frac{\cos(\Gamma\tau+\theta)}{\cos\theta-\sin(\Gamma\tau)-\sin\theta\cos(\Gamma\tau+\theta)}. \tag{36}$$

In view of Lemma 1, this gives rise to

$$\|G(s,\ \lambda)\|_2^2 = \phi(0) = \left(\frac{1}{2(\beta+\alpha p)|k|\lambda}\right)\frac{\cos(\Gamma\tau+\theta)}{\cos\theta-\sin(\Gamma\tau)-\sin\theta\cos(\Gamma\tau+\theta)}.$$

The result then follows by noting the identity

$$\cos\theta - \sin(\Gamma\tau) - \sin\theta\cos(\Gamma\tau+\theta) = \cos\theta\,(1-\sin(\Gamma\tau+\theta))\,.$$

■

While Theorem 2 exhibits an intricate dependence of the error performance on the agent and network characteristics, insights may still be gained by resorting to analysis of special cases. We make below a number of observations to this effect.

Remark 3. For a delay-free protocol, i.e., when $\tau = 0$, the expression (28) reduces to

$$E = \frac{1}{2(\beta+\alpha p)|k|}\sum_{i=2}^{N}\frac{\sigma_i^2}{\lambda_i(1-\sin\theta_i)} = \frac{1}{2}\sum_{i=2}^{N}\frac{1}{(\beta|k|\lambda_i - p)(1-\alpha|k|\lambda_i)}. \tag{37}$$

For simplicity, consider further $\alpha = 1$, $\beta = z$. Then,

$$E = \frac{1}{2z}\sum_{i=2}^{N}\frac{\sigma_i^2}{(|k|\lambda_i - (p/z))(1-|k|\lambda_i)},$$

thus exhibiting the negative effect of the nonminimum phase zero on the consensus performance. As p/z increases, i.e., when the nonminimum phase zero and the unstable pole of the agents become closer, the consensus performance is worsened. By a more in-depth inspection, one can assert that this statement holds in general. Indeed, since both Γ_i and θ_i increase monotonically with α, E decreases monotonically with α. This suggests that for agents without a finite nonminimum phase zero, that is, $\alpha = 0$, the error performance is better, exhibiting yet again the degrading effect of nonminimum phase zero. Note in particular that for integrator agents, i.e., when $\alpha = 0$, $p = 0$, then $\Gamma_i = \beta|k|\lambda_i$, $\theta_i = 0$. Consequently,

$$E = \frac{1}{2\beta|k|}\sum_{i=2}^{N}\frac{\cos(\beta|k|\lambda_i\tau)\sigma_i^2}{\lambda_i(1-\sin(\beta|k|\lambda_i\tau))}.$$

We note that this result was previously obtained in[42]. Interestingly, the expression in (28), while for considerably more sophisticated agent dynamics, retains its essential form.

Remark 4. By a direct calculation, we find that

$$\frac{dE}{d\tau} = \frac{1}{2(\beta+\alpha p)|k|}\sum_{i=2}^{N}\frac{\Gamma_i\sigma_i^2}{\lambda_i\cos\theta_i(1-\sin(\Gamma_i\tau+\theta_i))}.$$

This enables us to conclude that E increases monotonically with $\tau \in [0,\ \bar{\tau})$, demonstrating the effect of time delay on the consensus performance.

5 | ILLUSTRATIVE EXAMPLES

We now illustrate the preceding analytical results. Consider the MAS (8) with 6 agents, connected by an undirected network described by the Laplacian matrix

$$\mathcal{L} = \begin{bmatrix} 8 & -4 & 0 & -2 & -2 & 0 \\ -4 & 7 & -3 & 0 & 0 & 0 \\ 0 & -3 & 6 & 0 & 0 & -3 \\ -2 & 0 & 0 & 4 & 0 & -2 \\ -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & -2 & 0 & 5 \end{bmatrix}.$$

This matrix has an eigenvalue at the origin and other eigenvalues at $\lambda_2 = 1.6193$, $\lambda_3 = 3.7549$, $\lambda_4 = 4.5889$, $\lambda_5 = 9.3741$, and $\lambda_6 = 12.6628$. We first examine the effect of the unstable pole on the DCM. For this purpose, we fix the value of $\beta = 0.1$ and let the unstable pole p vary from 0 to 0.003. We also compare the cases $\alpha = 0$ and $\alpha = 2$. Fig. 1 shows that the DCM decreases rapidly even when p increases slightly. It also demonstrates that the presence of the nominimum phase zero reduces the DCM. Next, we examine the effect of the nonminimum phase zero. Fix $p = 0.8$ and select $\alpha = 1$. Note that by the condition (15), it is necessary that

$$\beta > \frac{\lambda_N}{\lambda_2}p = \frac{12.6628}{1.6193}(0.8) = 6.2559.$$

Hence, let $z = \beta$ vary in the interval $[6.4,\ 12]$. Fig. 2 shows that as z moves away further from the unstable pole p, the DCM increases.

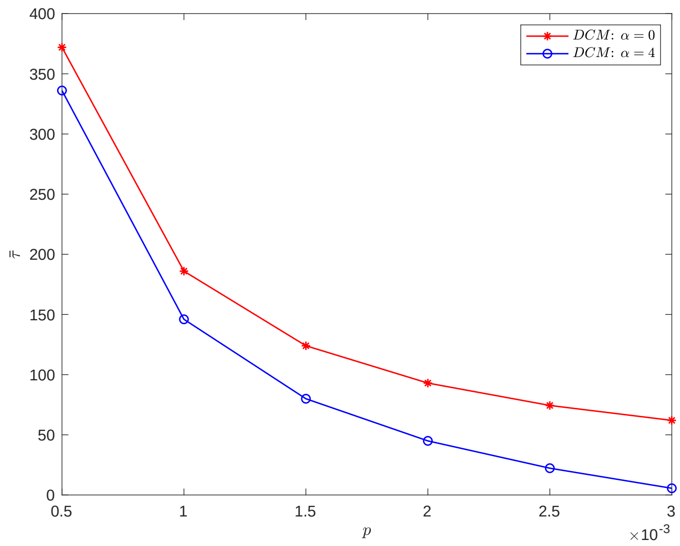

Figure 1 Delay consensus margin with different α: $\alpha = 0$ and $\alpha = 4$.

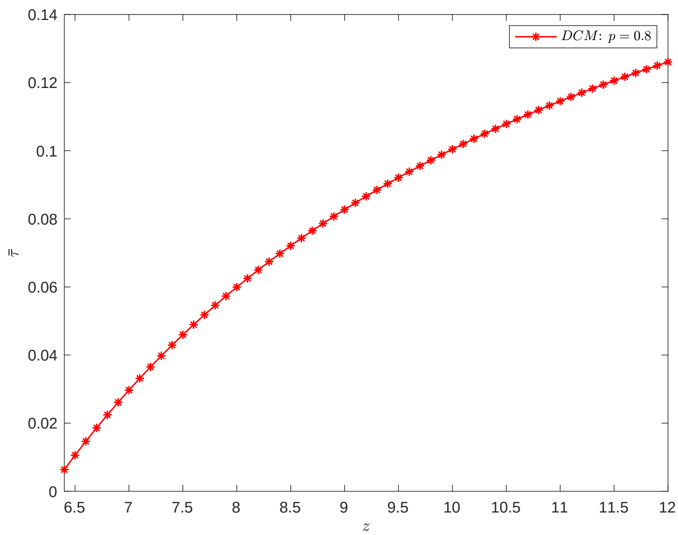

Figure 2 Delay consensus margin with different nonminimum phase zero:$p = 0.8$.

It is also of interest to see how the agents behave for different values of the delay parameter. Select $p = 0.15$, $\alpha = 2$, and $\beta = 5$. Thus, the agents have a nonminimum phase zero at $\beta/\alpha = 2.5$. With the coupling gain $k = -0.02$, the DCM given in (17) is computed as $\bar{\tau} = 0.7382$. Set the initial condition to be $x(0) = [-10, 5, 15, -15, 20, 1]^T$. For $\tau = 0.6 < \bar{\tau}$, Fig. 3 shows that the consensus error responses converge to zero, while for $\tau = 0.8 > \bar{\tau}$, Fig. 4 indicates that the consensus is not achieved.

Finally, we compute the consensus error performance. Select the same $p = 0.15$, $\alpha = 2$, and $\beta = 5$. Fig. 5 shows that E increases monotonically with τ. In particular, when τ approaches $\bar{\tau}$, E increases drastically. For purpose of comparison, we also plot E in the case of $\alpha = 0$. Note that in this latter case, the DCM is computed by (18), which yields $\bar{\tau} = 1.24$. The error performance is seen markedly better.

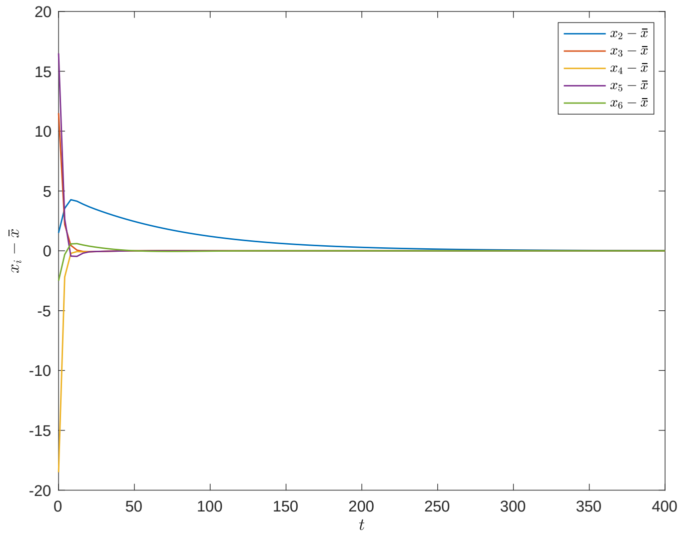

Figure 3 Consensus achieved at $\tau = 0.6$: Error responses.

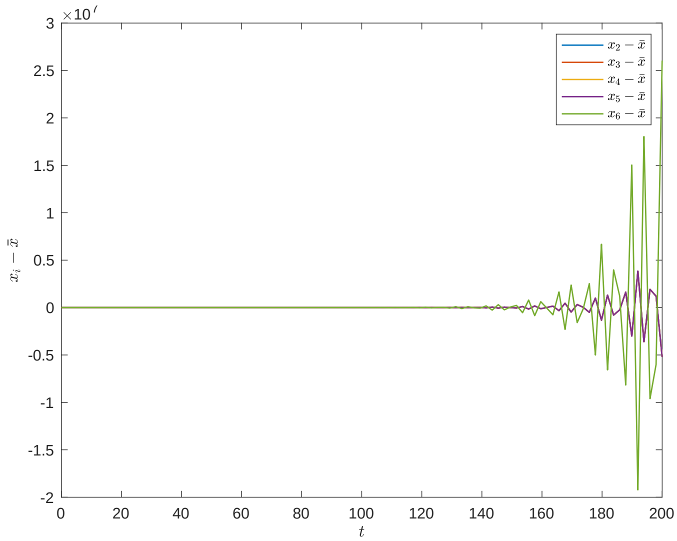

Figure 4 Consensus not achieved at $\tau = 0.8$: Error responses.

6 | CONCLUSION

In this paper we have addressed consensus robustness and performance problems for first-order unstable MASs, under delayed consensus output feedback subject to unknown time delay. We derived an analytical expression for the delay consensus margin, which serves a fundamental delay robustness measure, the largest range of delay so that consensus may be achieved robustly within that range. We also derived an explicit expression for the consensus error performance, which quantifies the disruptive effect of random nodal noises on consensus. The latter problem translates into the calculation of the $\mathcal{H}_2$ norm of a delay system. Both results exhibit useful insights into the constraints imposed by the agents' unstable pole and nonminiumum phase zero, as well as by the network topology and network delay on the consensus robustness and performance.

It should be pointed out, nonetheless, that our present development is restricted to networks described by an undirected graph and feedback protocols of an uniform delay. More general problems that incorporate, e.g., directed graphs and heterogeneous delays pose technical challenges and remain open. The uncertainties in the agent dynamics, likely to be heterogeneous and of a higher order, are also pertinent in the consensus robustness studies and require a more in-depth investigation.

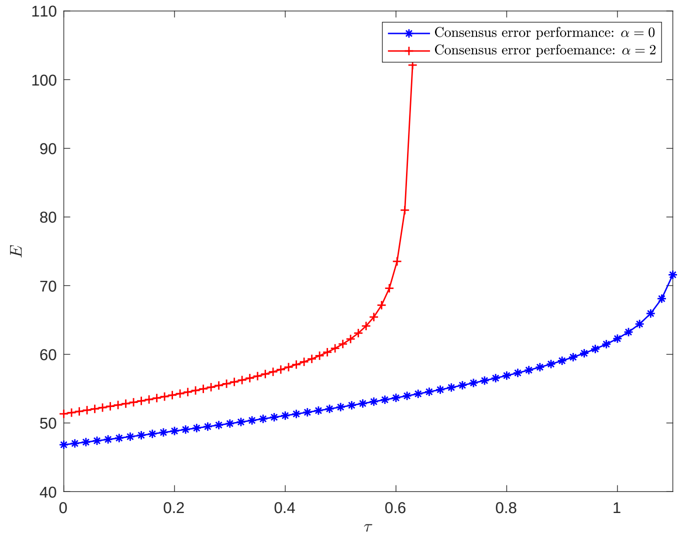

Figure 5 Consensus error performance as a function of τ.

References

1. Tahbaz-Salehi A, Jadbabaie A. Consensus over ergodic stationary graph processes. *IEEE Transactions on Automatic Control* 2009; 55(1): 225–230.

2. Olfati-Saber R, Murray RM. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control* 2004; 49(9): 1520–1533.

3. Tahbaz-Salehi A, Jadbabaie A. A necessary and sufficient condition for consensus over random networks. *IEEE Transactions on Automatic Control* 2008; 53(3): 791–795.

4. Lin P, Jia Y. Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies. *Automatica* 2009; 45(9): 2154–2158.

5. Cheng L, Hou ZG, Tan M, Wang X. Necessary and sufficient conditions for consensus of double-integrator multi-agent systems with measurement noises. *IEEE Transactions on Automatic Control* 2011; 56(8): 1958-1963.

6. Nuño E, Ortega R, Basañez L, Hill D. Synchronization of networks of nonidentical Euler-Lagrange systems with uncertain parameters and communication delays. *IEEE Transactions on Automatic Control* 2011; 56(4): 935-941.

7. Yang T, Roy S, Wan Y, Saberi A. Constructing consensus controllers for networks with identical general linear agents. *International Journal of Robust and Nonlinear Control* 2011; 21(11): 1237–1256.

8. Chopra N, Spong MW. On exponential synchronization of Kuramoto oscillators. *IEEE Transactions on Automatic Control* 2009; 54(2): 353–357.

9. Lin Z, Broucke M, Francis B. Local control strategies for groups of mobile autonomous agents. *IEEE Transactions on Automatic Control* 2004; 49(4): 622–629.

10. Shen B, Wang Z, Liu X. A stochastic sampled-data approach to distributed $\mathcal{H}_\infty$ filtering in sensor networks. *IEEE Transactions on Circuits and Systems I: Regular Papers* 2011; 58(9): 2237–2246.

11. Chen F, Chen J. Minimum-energy distributed consensus control of multiagent systems: A network approximation approach. *IEEE Transactions on Automatic Control* 2019; 65(3): 1144–1159.

12. Young GF, Scardovi L, Leonard NE. Robustness of noisy consensus dynamics with directed communication. In: Proceedings of the 2010 American Control Conference. IEEE. ; 2010: 6312–6317.

13. Siami M, Motee N. Fundamental limits and tradeoffs on disturbance propagation in linear dynamical networks. *IEEE Transactions on Automatic Control* 2016; 61(12): 4055-4062.

14. Somarakis C, Ghaedsharaf Y, Motee N. Time-delay origins of fundamental tradeoffs between risk of large fluctuations and network connectivity. *IEEE Transactions on Automatic Control* 2019; 64(9): 3571–3586.

15. Bamieh B, Jovanovic MR, Mitra P, Patterson S. Coherence in large-scale networks: Dimension-dependent limitations of local feedback. *IEEE Transactions on Automatic Control* 2012; 57(9): 2235–2249.

16. Ding Y, Peng H, Chen J. Consensus Error Performance of Linear Multi-Agent Systems. In: 2022 IEEE 61st Conference on Decision and Control (CDC). IEEE. ; 2022: 328–333.

17. Fitch K, Leonard NE. Information centrality and optimal leader selection in noisy networks. In: IEEE. ; 2013: 7510–7515.

18. Siami M, Bolouki S, Bamieh B, Motee N. Centrality measures in linear consensus networks with structured network uncertainties. *IEEE Transactions on Control of Network Systems* 2017; 5(3): 924–934.

19. Somarakis C, Ghaedsharaf Y, Motee N. Time-delay origins of fundamental tradeoffs between risk of large fluctuations and network connectivity. *IEEE Transactions on Automatic Control* 2019; 64(9): 3571–3586.

20. Trentelman HL, Takaba K, Monshizadeh N. Robust synchronization of uncertain linear multi-agent systems. *IEEE Transactions on Automatic Control* 2013; 58(6): 1511–1523.

21. Mei J, Ren W, Chen J. Distributed consensus of second-order multi-agent systems with heterogeneous unknown inertias and control gains under a directed graph. *IEEE Transactions on Automatic Control* 2015; 61(8): 2019–2034.

22. Wang B, Chen W, Zhang B. Semi-global robust tracking consensus for multi-agent uncertain systems with input saturation via metamorphic low-gain feedback. *Automatica* 2019; 103: 363–373.

23. Li X, Ma D, Hu X, Sun Q. Dynamic event-triggered control for heterogeneous leader-following consensus of multi-agent systems based on input-to-state stability. *International Journal of Control, Automation and Systems* 2020; 18(2): 293–302.

24. Azzollini IA, Yu W, Yuan S, Baldi S. Adaptive leader-follower synchronization over heterogeneous and uncertain networks of linear systems without distributed observer. *IEEE Transactions on Automatic Control* 2020; 66(4): 1925–1931.

25. Zelazo D, Bürger M. On the robustness of uncertain consensus networks. *IEEE Transactions on Control of Network Systems* 2015; 4(2): 170–178.

26. Li Z, Chen J. Robust consensus of linear feedback protocols over uncertain network graphs. *IEEE Transactions on Automatic Control* 2017; 62(8): 4251–4258.

27. Li Z, Chen J. Robust consensus for multi-agent systems communicating over stochastic uncertain networks. *SIAM Journal on Control and Optimization* 2019; 57(5): 3553–3570.

28. Ma D, Chen J, Lu R, Chen J, Chai T. Delay consensus margin of first-order multiagent systems with undirected graphs and PD protocols. *IEEE Transactions on Automatic Control* 2020; 66(9): 4192–4198.

29. Ma D, Chen J, Lu R, Chen J, Chai T. Delay effect on first-order consensus over directed graphs: Optimizing PID protocols for maximal robustness. *SIAM Journal on Control and Optimization* 2022; 60(1): 233–258.

30. Olfati-Saber R, Murray RM. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control* 2004; 49(9): 1520–1533.

31. Xu J, Zhang H, Xie L. Input delay margin for consensusability of multi-agent systems. *Automatica* 2013; 49(6): 1816–1820.

32. Münz U, Papachristodoulou A, Allgöwer F. Delay robustness in consensus problems. *Automatica* 2010; 46(8): 1252–1265.

33. Münz U, Papachristodoulou A, Allgöwer F. Delay robustness in non-identical multi-agent systems. *IEEE Transactions on Automatic Control* 2011; 57(6): 1597–1603.

34. Ma D. Delay range for consensus achievable by proportional and PD feedback protocols with time-varying delays. *IEEE Transactions on Automatic Control* 2021; 67(6): 3212–3219.

35. Godsil C, Royle GF. *Algebraic Graph Theory*. 207. Springer Science & Business Media . 2013.

36. Kharitonov V. *Time-Delay Systems: Lyapunov Functionals and Matrices*. Springer Science & Business Media . 2012.

37. Jarlebring E, Vanbiervliet J, Michiels W. Characterizing and computing the $\mathcal{H}_2$ norm of time-delay systems by solving the delay Lyapunov equation. *IEEE Transactions on Automatic Control* 2010; 56(4): 814–825.

38. Gu K, Chen J, Kharitonov VL. *Stability of Time-delay Systems*. Springer Science & Business Media . 2003.

39. Middleton RH, Miller DE. On the achievable delay margin using LTI control for unstable plants. *IEEE Transactions on Automatic Control* 2007; 52(7): 1194-1207.

40. Ma D, Chen J. Delay margin of low-order systems achievable by PID controllers. *IEEE Transactions on Automatic Control* 2019; 64(5): 1958–1973.

41. Boros G, Moll VH. Sums of arctangents and some formulas of Ramanujan. *SCIENTIA Series A: Mathematical Sciences* 2005; 10: 13–24.

42. Dezfulian S, Ghaedsharaf Y, Motee N. Performance analysis and optimal design of time-delay directed consensus networks. *IEEE Transactions on Control of Network Systems* 2021; 9(1): 197–209.

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