Dynamical analysis and numerical simulation of a stochastic influenza transmission model with human mobility and Ornstein-Uhlenbeck process

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Abstract

With the inevitable environmental perturbations and complex population movements, the analysis of troublesome influenza is harder to proceed. Studies about the epidemic mathematical models can not only forecast the development trend of influenza, but also have a beneficial influence on the protection of health and the economy. Motivated by this, a stochastic influenza model incorporating human mobility and the Ornstein-Uhlenbeck process is established in this paper. Based on the existence of the unique global positive solution, we obtain sufficient conditions for influenza extinction and persistence, which are related to the basic reproduction number in the corresponding deterministic model. Notably, the analytical expression of the probability density function of stationary distribution near the quasi-endemic equilibrium is obtained by solving the challenging Fokker–Planck equation. Finally, numerical simulations are performed to support the theoretical conclusions, and the effect of main parameters and environmental perturbations on influenza transmission are also investigated.

Keywords: Susceptible-Infectious-mobility model; Ornstein-Uhlenbeck process; Stationary distribution and extinction; Probability density function.

1. Introduction

Influenza is a serious public health problem which is caused by viruses that undergo continuous antigenic change and possess an animal reservoir [1]. Since viruses are prone to mutation, strong infectivity, widespread susceptibility and high incidence, it has caused many explosive epidemics around the world and gained global concern. The World Health Organization estimates that seasonal influenza causes 3 to 5 million severe cases and 29 to 65 million global deaths each year. Spanish influenza [2, 3] is the deadliest infectious disease in human history, infecting about 1 billion people worldwide and killing at least 25 million in 1918-1919 (the world population was about 1.7 billion in that period). The Hong Kong influenza outbreak [4] in 1968 was the first pandemic caused by the H3N2 influenza virus, which was derived from an antigenic transformation of the H2N2 virus that caused the "Asian influenza" [5] pandemic in 1957-1958. About 15 percent of the local population became infected and gradually spread to Singapore, Thailand, Japan, India and Australia in August. At the end of the same year, it rapidly appeared in North America. These alarming statistics remind us of the urgency of scientific theoretical analysis of influenza.

Since Kermack and Mckendrick [6] creatively constructed the classical SIR (Susceptible-Infectious-Recover) compartment epidemic model to investigate the Black Death, plenty of mathematical models are used to describe and analyze the transmission mechanism of troublesome epidemics. Considering the seasonality of the influenza pandemic, J. M. Tchuenche et al. [7] formulated an SVITR epidemic model with vaccination and treatment. They mainly focused on controlling the disease with a possible minimal cost and side effects. J. Lucchetti et al. [8] proposed an avian influenza model to deal with several complicated situations, i.e., some low pathogenic avian influenza viruses will become high pathogenic strains after transmission to domestic birds that can infect humans and create potential conditions for another influenza pandemic. As more realistic factors are considered, many novel mathematical models that fit reality are built and analyzed [9, 10, 11, 12].

Classical influenza models do not account for behavioral change. For example, the transmission rate is supposed to be constant, which implies individuals do not adapt their contact behavior during epidemics. It seems unreasonable because the diversification of multimedia forms makes people more sensitive to external

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information. At the same time, the acceleration of the process of world economic integration and convenient transportation enable people to move wider and faster. The susceptible population would take appropriate preventive measures against disease transmission when they feel threatened, such as wearing masks, keeping social distance and reducing contact with others. These will feed back to alter epidemic dynamics, so it is better to add human behavior effect to mathematical models. J. M. Epstein [13] et al. studied the coupled contagion dynamics of fear and disease. In [14], A. d'Onofrio and P. Manfredi assumed that social contact rate is a function of the available information about disease prevalence and they concluded that social behavior change alone may trigger sustained oscillations. Wang [15] modeled adaptive behavior in influenza transmission and proposed the following mathematical system

$$\begin{cases} \mathrm{d}S(t) = \left(\Lambda - \mu S - \frac{\bar{\beta} m I S}{1+hI}\right)\mathrm{d}t, \\ \mathrm{d}I(t) = \left(\frac{\bar{\beta} m I S}{1+hI} - (\mu+\gamma) I\right)\mathrm{d}t, \\ \mathrm{d}m(t) = m\left(b - am - \frac{\alpha I}{1+hI}\right)\mathrm{d}t, \end{cases} \tag{1.1}$$

where S, I and m donate the susceptible individuals, infectious individuals and human mobility, respectively, Λ is the recruitment constant of the susceptible class, μ represents the birth and death rate, γ is the sum of recovery and treatment rate, $\bar{\beta}$ and $\frac{b}{a}$ denote the transmission rate and the social capacity of population mobility in the context of economics, respectively, α and h are positive constants that determine the reduction of mobility caused by influenza. For the deterministic model (1.1), Wang mainly obtained several conclusions as follows

- The basic reproduction number is $R_0 = \frac{\Lambda \bar{\beta} b}{\mu a(\mu+\gamma)}$.
- There always exist semi-trivial equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0)$ and disease-free equilibrium $E_1 = (\frac{\Lambda}{\mu}, 0, \frac{b}{a})$. Furthermore, if $R_0 > 1$, then system (1.1) has a unique endemic equilibrium $E_2 = (S^*, I^*, m^*)$.
- E_1 is globally stable if $R_0 < 1$. Let $R_0 > 1$, then E_2 is locally asymptotically stable.

In the real environment, there are many uncertainties that would critically perturb the transmission of influenza, such as ambient temperature, control strategies, travel of populations, and so on. Therefore, adding randomness to deterministic models is close to reality. Meanwhile, with the continuous improvement of the stochastic differential equation [16] and other basic disciplines, it provides an effective theoretical basis for the creation and property analysis of stochastic models. For example, Meng et al. [17] studied a stochastic eco-epidemiological model with time delay and they got sufficient conditions of permanence in mean or extinction for the ecological populations. Since the environment may suffer sudden shocks, Yuan and Zhao [18] dealt with the stochastic epidemic model incorporating jump-diffusion infection force.

Environmental perturbations can be introduced to mathematical models by supposing that the parameters change with time [19]. Under the assumption that the transmission rate $\bar{\beta}$ is a linear function of white noise $\bar{\beta} + \sigma \mathrm{d}B(t)$, Cai et al. [20] proposed the following stochastic influenza model corresponding to the deterministic model (1.1) and focused on its dynamic behaviors

$$\begin{cases} \mathrm{d}S(t) = \left(\Lambda - \mu S - \frac{\bar{\beta} m I S}{1+hI}\right)\mathrm{d}t - \frac{\sigma m I S}{1+hI}\mathrm{d}B(t), \\ \mathrm{d}I(t) = \left(\frac{\bar{\beta} m I S}{1+hI} - (\mu+\gamma) I\right)\mathrm{d}t + \frac{\sigma m I S}{1+hI}\mathrm{d}B(t), \\ \mathrm{d}m(t) = m\left(b - am - \frac{\alpha I}{1+hI}\right)\mathrm{d}t, \end{cases}$$

where $B(t)$ is the standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), σ^2 represents the intensity of the Brownian motion. However, after studying E. Allen [21] and other scholars' [22, 23] researches, we know that the average value of modified transmission rate over an arbitrary time interval $[0, T]$ satisfies

$$\frac{1}{T}\int_0^T \beta(\tau)\mathrm{d}\tau \sim \mathcal{N}\left(\bar{\beta}, \frac{\sigma^2}{T}\right),$$

in which $\mathcal{N}$ denotes the normal distribution. This causes a unreasonable result that the variance $\frac{\sigma^2}{T} \to +\infty$ as $T \to 0$, meaning that the modified parameter $\beta(t)$ will drastically change in a short period which conflicts with continuous environment. To deal with this shortcoming, various stochastic models adapt the assumption that the modified parameter satisfies the mean-reversing process [24, 25, 26, 27] instead of the linear functions of white noise. Inspired by these brilliant ideas, in this paper, we suppose that the transmission rate $\bar{\beta}$ is the log-normal Ornstein-Uhlenbeck process, that is to say

$$\mathrm{d}\log\beta(t) = \theta\left(\log\bar{\beta} - \log\beta\right)\mathrm{d}t + \sigma\mathrm{d}B(t),$$

where $\theta > 0$ represents the reversion speed and $\sigma > 0$ denotes the volatility intensity. What is more important, some calculations lead to

$$\frac{1}{T}\int_0^T \beta(\tau)\mathrm{d}\tau \sim \mathcal{N}\left(\bar{\beta}, \frac{\sigma^2 T}{3} + \mathcal{O}(T^2)\right),$$

in which $\mathcal{O}(T^2)$ is the higher order infinitesimal of T^2. It is obvious that the variance $\frac{\sigma^2 T}{3} + \mathcal{O}(T^2) \to 0$ as $T \to 0$. In this sense, we construct the following stochastic influenza model with human mobility and log-normal Ornstein-Uhlenbeck process

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - \mu S - \frac{\beta m I S}{1+hI}\right]\mathrm{d}t, \\ \mathrm{d}I(t) = \left[\frac{\beta m I S}{1+hI} - (\mu+\gamma)I\right]\mathrm{d}t, \\ \mathrm{d}m(t) = \left[m\left(b - am - \frac{\alpha I}{1+hI}\right)\right]\mathrm{d}t, \\ \mathrm{d}\log\beta(t) = \theta\left(\log\bar{\beta} - \log\beta\right)\mathrm{d}t + \sigma\mathrm{d}B(t). \end{cases} \tag{1.2}$$

The dynamical properties of this stochastic system (1.2) are mainly discussed in the rest content.

The structure of this paper is arranged as follows. Section 2 verifies the existence and uniqueness of the global positive solution of the stochastic model (1.2). Section 3 deals with sufficient conditions for the extinction and persistence of influenza. With the existence of stationary distribution, we further calculate the analytic expression of the probability density function around the quasi-endemic equilibrium in Section 4. Finally, in Section 5, several numerical examples are provided to illustrate the theoretical counterparts.

2. Solution of stochastic system (1.2)

From the perspective of biology, because S, I and m denote the number of susceptible population, infectious population and mobility intensity, respectively, and the transmission rate β satisfies the log-normal OU process, they should be non-negative. Therefore, the existence of a unique positive global solution is needed before we investigate the dynamical behavior of the stochastic influenza system (1.2). We will verify the existence and uniqueness of a positive global solution to the stochastic system (1.2) with any initial value.

Lemma 2.1. *(Itô's formula [16]) Suppose that $x(t)$ is an n-dimensional Itô's process on $t \geq 0$ with the stochastic differential*

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}B(t),$$

where $B(t) = (B_1(t), \cdots, B_m(t))^T$, $f \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^n)$ and $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{n\times m})$. If $V \in \mathbb{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, then $V(x(t), t)$ is a real-valued Itô's process and its stochastic differential is given by

$$\begin{aligned}\mathrm{d}V(x(t), t) =& \left[V_t(x(t), t) + V_x(x(t), t)f(t) + \frac{1}{2}trace\left(g^T(t)V_{xx}(x(t), t)g(t)\right)\right]\mathrm{d}t + V_x(x(t), t)g(t)\mathrm{d}B(t) \\ =& \mathcal{L}V(x(t), t)\mathrm{d}t + V_x(x(t), t)g(t)\mathrm{d}B(t), \ a.s.,\end{aligned}$$

in which $\mathcal{L}V(x(t), t)$ denotes the differential operator and

$$V_t = \frac{\partial V}{\partial t}, \ V_x = \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n}\right), \ V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Theorem 2.1. *For any initial value $(S(0), I(0), m(0), \beta(0)) \in \mathbb{R}_+^4$, the stochastic system (1.2) has a unique positive global solution $(S(t), I(t), m(t), \beta(t))$ $(t \geq 0)$ almost sure (a.s.). Furthermore, if $S(0) + I(0) < \frac{\Lambda}{\mu}$ and $m(0) < \frac{b}{a}$, then the system (1.2) has an invariant set Ω as follows*

$$\Omega = \left\{(S, \ I, \ m, \ \beta) \in \mathbb{R}_+^4 | S + I < \frac{\Lambda}{\mu}, \ m < \frac{b}{a}\right\}.$$

Proof. The detailed proof can refer to the methods in [28]. Here, we only represent the key of these proofs, i.e., the construction of a non-negative Lyapunov C^2-function. Define a C^2-function $W(S, I, m, \beta)$: $\mathbb{R}_+^4 \to \mathbb{R}_+$ by

$$W(S, I, m, \beta) = S - 1 - \log S + I - 1 - \log I + m - 1 - \log m + \beta - 1 - \log \beta.$$

The nonnegativity of W can be deduced from the inequality $x - 1 \geq \log x$ $(x > 0)$. Applying Itô's formula to the stochastic system (1.2) follows

$$\begin{aligned}
\mathcal{L}(-\log S) &= -\frac{\Lambda}{S} + \frac{\beta I m}{1 + hI} + \mu,\ \mathcal{L}(-\log I) = -\frac{\beta S m}{1 + hI} + \mu + \gamma, \\
\mathcal{L}(-\log m) &= am + \frac{\alpha I}{1 + hI} - b,\ \mathcal{L}\beta = \beta(\theta \log \bar{\beta} - \theta \log \beta + \frac{1}{2}\sigma^2).
\end{aligned} \tag{2.1}$$

By (2.1) and Itô's formula, we have

$$\begin{aligned}
\mathcal{L}W =& \Lambda - \mu(S + I) - \gamma I + m\left(b - am - \frac{\alpha I}{1 + hI}\right) - \frac{\Lambda}{S} + \frac{\beta m I}{1 + hI} + \mu - \frac{\beta m S}{1 + hI} \\
&+ (\mu + \gamma) + am + \frac{\alpha I}{1 + hI} - b + \beta\left(\theta \log \bar{\beta} - \theta \log \beta + \frac{1}{2}\sigma^2\right) - \theta(\log \bar{\beta} - \log \beta) \\
\leq& \Lambda + 2\mu + \gamma + (a + b)m + \frac{\beta m I}{1 + hI} + \frac{\alpha I}{1 + hI} + \beta\left(\theta \log \bar{\beta} - \theta \log \beta + \frac{1}{2}\sigma^2\right) - \theta(\log \bar{\beta} - \log \beta).
\end{aligned} \tag{2.2}$$

It is obvious that

$$\frac{I}{1 + hI} \leq \frac{1}{h}. \tag{2.3}$$

From the third equation in the (1.2), it shows that

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = m\left(b - am - \frac{\alpha I}{1 + hI}\right) \leq m\,(b - am)\,,$$

which means

$$m(t) \leq \begin{cases} m(0), & \text{if } m(0) \geq \frac{b}{a}, \\ \frac{b}{a}, & \text{if } m(0) < \frac{b}{a}. \end{cases} \tag{2.4}$$

Let $K = \max\left\{m(0),\ \frac{b}{a}\right\}$ and combining (2.3), then the inequality (2.2) becomes

$$\begin{aligned}
\mathcal{L}W \leq& \Lambda + 2\mu + \gamma + (a + b)m + \frac{\beta m I}{1 + hI} + \frac{\alpha I}{1 + hI} + \beta\left(\theta \log \bar{\beta} - \theta \log \beta + \frac{1}{2}\sigma^2\right) - \theta(\log \bar{\beta} - \log \beta) \\
\leq& \Lambda + 2\mu + \gamma + \frac{\alpha}{h} + (a + b)K + f(\beta),
\end{aligned}$$

where

$$f(\beta) = \frac{K\beta}{h} + \beta\left(\theta \log \bar{\beta} - \theta \log \beta + \frac{1}{2}\sigma^2\right) - \theta(\log \bar{\beta} - \log \beta).$$

Notably, $f(\beta) \to -\infty$ as $\beta \to 0^+$ or $\beta \to +\infty$, which implies there exists $\sup_{\beta \in \mathbb{R}_+} f(\beta)$. Hence, we have

$$\begin{aligned}
\mathcal{L}W \leq& \Lambda + 2\mu + \gamma + \frac{\alpha}{h} + (a + b)K + f(\beta) \\
\leq& \Lambda + 2\mu + \gamma + \frac{\alpha}{h} + (a + b)K + \sup_{\beta \in \mathbb{R}_+} f(\beta) \\
\triangleq& C,
\end{aligned}$$

where C is a positive constant. The proof is completed. $\square$

Remark 2.1. *Similar to the (2.4), we can also get*

$$S(t)+I(t)\leq\begin{cases}S(0)+I(0), & \text{if } S(0)+I(0)\geq\frac{\Lambda}{\mu},\\ \frac{\Lambda}{\mu}, & \text{if } S(0)+I(0)<\frac{\Lambda}{\mu},\end{cases}$$

which means the stochastic system (1.2) has the invariant set $\Omega=\left\{(S,\ I,\ m,\ \beta)\in\mathbb{R}_+^4|S+I<\frac{\Lambda}{\mu},\ m<\frac{b}{a}\right\}$ *with the initial value satisfying* $S(0)+I(0)<\frac{\Lambda}{\mu}$ *and* $m(0)<\frac{b}{a}$.

From now on, unless otherwise stated, we always assume that $S(0)+I(0)<\frac{\Lambda}{\mu}$ *and* $m(0)<\frac{b}{a}$*, then we investigate the stochastic system (1.2) in the invariant set* Ω.

3. Extinction and persistence

In this section, we will investigate the tendency of influenza transmission, and give sufficient conditions for the extinction and persistence of the influenza. By constructing several appropriate Lyapunov functions, we obtain two critical values

$$R_0^e=R_0e^{\frac{\sigma^2}{4\theta}},\ R_0^s=R_0\left[1-3\left(e^{\frac{\sigma^2}{9\theta}}-2e^{\frac{\sigma^2}{36\theta}}+1\right)^{\frac{1}{2}}\right],$$

which are completely determined by the basic reproduction numbers R_0 and environmental perturbations.

Definition 3.1. *[29] (i) x(t) is said to be exponential extinct if* $\lim_{t\to+\infty}\frac{\log x(t)}{t}<0$ *a.s..*

(ii) x(t) is said to be strongly persistent in the mean if $\liminf_{t\to+\infty}\frac{1}{t}\int_0^t x(\tau)\mathrm{d}\tau>0$ *a.s..*

Lemma 3.1. *[22, 25, 30] If* $\beta(t)$ *satisfies the log-normal Ornstein-Uhlenbeck process*

$$\mathrm{d}\log\beta(t)=\theta\left(\log\bar{\beta}-\log\beta\right)\mathrm{d}t+\sigma\mathrm{d}B(t),$$

then $\beta(t)$ *is ergodic and* $\log\beta(t)$ *has the following probability density function as* $t\to\infty$

$$\zeta(y)=\frac{\sqrt{\theta}}{\sqrt{\pi}\sigma}e^{-\theta\left(\frac{y-\log\bar{\beta}}{\sigma}\right)^2}.$$

Theorem 3.1. *Assume that* $R_0^e<1$*, then the influenza of the stochastic system (1.2) will exponentially die out a.s.. To be specific, we have*

$$\limsup_{t\to+\infty}\frac{\log I(t)}{t}\leq(\mu+\gamma)(R_0^e-1)<0\ a.s..$$

Proof. According to the invariant set Ω, we have

$$\mathrm{d}(\log I)=\frac{\beta mS}{1+hI}-(\mu+\gamma)\leq\beta mS-(\mu+\gamma)\leq\frac{\Lambda b}{\mu a}\beta-(\mu+\gamma). \tag{3.1}$$

Integrating (3.1) from 0 to t and dividing t on both sides give the following inequality

$$\frac{\log I(t)-\log I(0)}{t}\leq\frac{\Lambda b}{\mu a}\frac{1}{t}\int_0^t\beta(\tau)\mathrm{d}\tau-(\mu+\gamma). \tag{3.2}$$

With the lemma 3.1, we obtain the following equation by letting $t\to+\infty$,

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t\beta(\tau)\mathrm{d}\tau=\int_{-\infty}^{+\infty}e^y\frac{\sqrt{\theta}}{\sqrt{\pi}\sigma}e^{-\theta(\frac{y-\log\bar{\beta}}{\sigma})^2}\mathrm{d}y=\bar{\beta}e^{\frac{\sigma^2}{4\theta}}\ a.s.. \tag{3.3}$$

Assuming $t \to +\infty$ and substituting (3.3) into (3.2), we have

$$
\begin{aligned}
\limsup_{t\to+\infty} \frac{\log I(t)}{t} \leq & \frac{\Lambda b}{\mu a} \lim_{t\to+\infty} \frac{1}{t}\int_0^t \beta(\tau)\mathrm{d}\tau - (\mu+\gamma) \\
= & \frac{\Lambda b \bar{\beta}}{\mu a} e^{\frac{\sigma^2}{4\theta}} - (\mu+\gamma) \\
= & (\mu+\gamma)(R_0^e - 1) \ a.s.,
\end{aligned}
\tag{3.4}
$$

where

$$
R_0^e = R_0 e^{\frac{\sigma^2}{4\theta}}.
$$

Obviously, if $R_0^e < 1$, then $\limsup_{t\to+\infty} \frac{\log I(t)}{t} \leq (\mu+\gamma)(R_0^e - 1) < 0$, this indicates that the influenza of the stochastic system (1.2) will exponentially become extinct a.s.. The proof ends. ☐

Theorem 3.1 gives the condition of influenza extinction. For the comprehensive study, we investigate another concerned issue, i.e., the persistence of influenza.

Theorem 3.2. *If $R_0^s > 1$, then the infectious population $I(t)$ will strongly persistent in the mean a.s.. More precisely,*

$$
\liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(\tau)\mathrm{d}\tau \geq \frac{2ac_5(\mu+\gamma)(R_0^s - 1)}{bc_4} \ a.s..
$$

The value of c_4 and c_5 are determined in the later proof.

Proof. Define the C^2-Lyapunov function V_1: $\Omega \to \mathbb{R}$ as

$$
V_1 = -\log I - c_1 \log S + c_2 I,
$$

where c_1 and c_2 are determined in (3.9). Let $c_3 = \frac{c_2(\mu+\gamma)}{h}$, then applying Itô's formula to V_1 reads

$$
\begin{aligned}
\mathcal{L}V_1 = & -\frac{\beta m S}{1+hI} + \mu + \gamma - c_1\frac{\Lambda}{S} + c_1\frac{\beta m I}{1+hI} + c_1\mu + c_2\frac{\beta m I S}{1+hI} - c_2(\mu+\gamma)I - \frac{c_2(\mu+\gamma)}{h} + \frac{c_2(\mu+\gamma)}{h} \\
= & -\frac{\beta m S}{1+hI} + \mu + \gamma - c_1\frac{\Lambda}{S} + c_1\frac{\beta m I}{1+hI} + c_1\mu + c_2\frac{\beta m I S}{1+hI} - c_3(1+hI) + c_3.
\end{aligned}
$$

Combining the invariant set Ω, we obtain

$$
\begin{aligned}
\mathcal{L}V_1 = & -\frac{\beta m S}{1+hI} - c_1\frac{\Lambda}{S} - c_3(1+hI) + c_1\mu + c_3 + \mu + \gamma + c_1\frac{\beta m I}{1+hI} + c_2\frac{\beta m I S}{1+hI} \\
\leq & -3\sqrt[3]{c_1 c_3 \Lambda \beta m} + c_1\mu + c_3 + \mu + \gamma + \left(c_1 + \frac{c_2\Lambda}{\mu}\right)\beta m I \\
\leq & -3\sqrt[3]{c_1 c_3 \Lambda \bar{\beta} m} + c_1\mu + c_3 + \mu + \gamma + \left(c_1 + \frac{c_2\Lambda}{\mu}\right)\beta m I + 3\sqrt[3]{\frac{b}{a} c_1 c_3 \Lambda} |\beta^{\frac{1}{3}} - \bar{\beta}^{\frac{1}{3}}|.
\end{aligned}
\tag{3.5}
$$

From the inequality

$$
\mathcal{L}\left(-\frac{1}{b}\log m\right) \leq \frac{a}{b}m + \frac{\alpha}{b}I - 1,
$$

one can get

$$
\mathcal{L}\left(-\frac{1}{b}\log m + \frac{\alpha}{(\mu+\gamma)b}I\right) \leq \frac{a}{b}m + \frac{\Lambda\alpha}{b\mu(\mu+\gamma)}\beta m I - 1. \tag{3.6}
$$

On the other hand, using Itô's formula obtains

$$
\mathcal{L}\left(\frac{2a}{3b^2}m\right) = \frac{2a}{3b^2}\left(bm - am^2 - \frac{\alpha m I}{1+hI}\right) \leq \frac{2}{3}\left[\frac{a}{b}m\left(1 - \frac{a}{b}m\right)\right]. \tag{3.7}
$$

Combining (3.6) and (3.7), we can define a C^2-Lyapunov function V_2: $\Omega \to \mathbb{R}$ by

$$V_2 = \frac{2a}{3b^2}m - \frac{1}{b}\log m + \frac{\alpha}{(\mu+\gamma)b}I,$$

and its differential operator satisfies the following inequality

$$\begin{aligned}\mathcal{L}V_2 \leq & \frac{a}{b}m + \frac{2a}{3b}m\left(1 - \frac{a}{b}m\right) + \frac{\Lambda\alpha}{b\mu(\mu+\gamma)}\beta m I - 1 \\ \leq & \sqrt[3]{\frac{a}{b}m} + \frac{\Lambda\alpha}{b\mu(\mu+\gamma)}\beta m I - 1,\end{aligned} \tag{3.8}$$

in which the last inequality sign is deduces by $x + \frac{2}{3}x(1-x) \leq \sqrt[3]{x}$ $(x > 0)$. Set

$$c_1\mu = \frac{c_2(\mu+\gamma)}{h} = c_3 = \frac{\Lambda\bar{\beta}b}{\mu a}, \tag{3.9}$$

and define the function

$$V_3 = V_1 + 3\sqrt[3]{\frac{b}{a}c_1c_3\Lambda\bar{\beta}}V_2.$$

Applying Itô's formula again, we get

$$\begin{aligned}\mathcal{L}V_3 \leq & -3\sqrt[3]{\frac{b}{a}c_1c_3\Lambda\bar{\beta}} + c_1\mu + c_3 + \mu + \gamma + c_4\beta m I + 3\sqrt[3]{\frac{b}{a}c_1c_3\Lambda}|\beta^{\frac{1}{3}} - \bar{\beta}^{\frac{1}{3}}| \\ = & -\frac{\Lambda\bar{\beta}b}{\mu a} + (\mu+\gamma) + c_4\beta m I + 3\sqrt[3]{\frac{b}{a}c_1c_3\Lambda}|\beta^{\frac{1}{3}} - \bar{\beta}^{\frac{1}{3}}| \\ = & -(\mu+\gamma)(R_0 - 1) + c_4\beta m I + 3\sqrt[3]{\frac{b}{a}c_1c_3\Lambda\bar{\beta}}\left(e^{\frac{\sigma^2}{9\theta}} - 2e^{\frac{\sigma^2}{36\theta}} + 1\right)^{\frac{1}{2}} \\ & + 3\sqrt[3]{\frac{b}{a}c_1c_3\Lambda}\left[|\beta^{\frac{1}{3}} - \bar{\beta}^{\frac{1}{3}}| - \bar{\beta}^{\frac{1}{3}}\left(e^{\frac{\sigma^2}{9\theta}} - 2e^{\frac{\sigma^2}{36\theta}} + 1\right)^{\frac{1}{2}}\right] \\ = & -(\mu+\gamma)(R_0^s - 1) + c_4\beta m I + \frac{3\Lambda b}{\mu a}\bar{\beta}^{\frac{2}{3}}\left[|\beta^{\frac{1}{3}} - \bar{\beta}^{\frac{1}{3}}| - \bar{\beta}^{\frac{1}{3}}\left(e^{\frac{\sigma^2}{9\theta}} - 2e^{\frac{\sigma^2}{36\theta}} + 1\right)^{\frac{1}{2}}\right],\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}c_4 = & c_1 + \frac{c_2\Lambda}{\mu} + \frac{3\Lambda\alpha}{b\mu(\mu+\gamma)}\sqrt[3]{\frac{b}{a}c_1c_3\Lambda}, \\ R_0^s = & R_0 - \frac{3\sqrt[3]{\frac{b}{a}c_1c_3\Lambda\bar{\beta}}\left(e^{\frac{\sigma^2}{9\theta}} - 2e^{\frac{\sigma^2}{36\theta}} + 1\right)^{\frac{1}{2}}}{\mu+\gamma} \\ = & R_0 - 3R_0\left(e^{\frac{\sigma^2}{9\theta}} - 2e^{\frac{\sigma^2}{36\theta}} + 1\right)^{\frac{1}{2}} \\ = & R_0\left[1 - 3\left(e^{\frac{\sigma^2}{9\theta}} - 2e^{\frac{\sigma^2}{36\theta}} + 1\right)^{\frac{1}{2}}\right].\end{aligned} \tag{3.11}$$

According to the inequality $x \leq ax^2 + \frac{1}{4a}(a > 0)$, there is a positive constant $c_5 = \frac{\mu a(\mu+\gamma)(R_0^s - 1)}{2c_4\Lambda b\bar{\beta}^2 e^{\frac{\sigma^2}{\theta}}}$ such that

$$\begin{aligned}\beta m I \leq & \left(c_5\beta^2 + \frac{1}{4c_5}\right)mI \leq \frac{c_5\Lambda b}{\mu a}\beta^2 + \frac{1}{4c_5}mI \\ = & \frac{c_5\Lambda b}{\mu a}\bar{\beta}^2 e^{\frac{\sigma^2}{\theta}} + \frac{1}{4c_5}mI + \frac{c_5\Lambda b}{\mu a}\left(\beta^2 - \bar{\beta}^2 e^{\frac{\sigma^2}{\theta}}\right) \\ = & \frac{(\mu+\gamma)(R_0^s - 1)}{2c_4} + \frac{1}{4c_5}mI + \frac{c_5\Lambda b}{\mu a}\left(\beta^2 - \bar{\beta}^2 e^{\frac{\sigma^2}{\theta}}\right).\end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.10), then it is transmitted into

$$\begin{aligned}\mathcal{L}V_3 \leq & -(\mu+\gamma)(R_0^s-1)+\frac{(\mu+\gamma)(R_0^s-1)}{2}+\frac{c_4}{4c_5}mI+\frac{c_4c_5\Lambda b}{\mu a}\left(\beta^2-\bar{\beta}^2e^{\frac{\sigma^2}{\theta}}\right)\\ &+\frac{3\Lambda b}{\mu a}\bar{\beta}^{\frac{2}{3}}\left[|\beta^{\frac{1}{3}}-\bar{\beta}^{\frac{1}{3}}|-\bar{\beta}^{\frac{1}{3}}\left(e^{\frac{\sigma^2}{9\theta}}-2e^{\frac{\sigma^2}{36\theta}}+1\right)^{\frac{1}{2}}\right]\\ =&-\frac{1}{2}(\mu+\gamma)(R_0^s-1)+\frac{c_4}{4c_5}mI+F(\beta),\end{aligned}\tag{3.13}$$

in which

$$F(\beta)=\frac{c_4c_5\Lambda b}{\mu a}\left(\beta^2-\bar{\beta}^2e^{\frac{\sigma^2}{\theta}}\right)+\frac{3\Lambda b}{\mu a}\bar{\beta}^{\frac{2}{3}}\left[|\beta^{\frac{1}{3}}-\bar{\beta}^{\frac{1}{3}}|-\bar{\beta}^{\frac{1}{3}}\left(e^{\frac{\sigma^2}{9\theta}}-2e^{\frac{\sigma^2}{36\theta}}+1\right)^{\frac{1}{2}}\right].$$

Since $m<\frac{b}{a}$, the inequality (3.13) can become

$$\mathcal{L}V_3\leq -\frac{1}{2}(\mu+\gamma)(R_0^s-1)+\frac{c_4b}{4c_5a}I+F(\beta).\tag{3.14}$$

Integrating (3.14) from 0 to t and dividing by t on both sides, then we have

$$\frac{bc_4}{4ac_5}\frac{1}{t}\int_0^t I(\tau)\mathrm{d}\tau\geq\frac{1}{2}(\mu+\gamma)(R_0^s-1)+\frac{V_3(t)-V_3(0)}{t}-\frac{1}{t}\int_0^t F(\beta(\tau))\mathrm{d}\tau.\tag{3.15}$$

From the lemma 3.1 and the Hölder's inequality [16], we get that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t\beta^2(\tau)\mathrm{d}\tau=\int_{-\infty}^{+\infty}e^{2y}\frac{\sqrt{\theta}}{\sqrt{\pi}\sigma}e^{-\theta(\frac{y-\log\bar{\beta}}{\sigma})^2}\mathrm{d}y=\bar{\beta}^2e^{\frac{\sigma^2}{\theta}}\ a.s.,$$

$$\begin{aligned}\limsup_{t\to+\infty}\frac{1}{t}\int_0^t|\beta^{\frac{1}{3}}(\tau)-\bar{\beta}^{\frac{1}{3}}|\mathrm{d}\tau\leq&\limsup_{t\to+\infty}\frac{1}{t}\left(\int_0^t1^2\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t\left(\beta^{\frac{1}{3}}(\tau)-\bar{\beta}^{\frac{1}{3}}\right)^2\mathrm{d}\tau\right)^{\frac{1}{2}}\\ =&\lim_{t\to+\infty}\frac{1}{t}\left(\int_0^t1^2\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_0^t\left(\beta^{\frac{1}{3}}(\tau)-\bar{\beta}^{\frac{1}{3}}\right)^2\mathrm{d}\tau\right)^{\frac{1}{2}}\\ =&\lim_{t\to\infty}\left(\frac{1}{t}\int_0^t(\beta^{\frac{1}{3}}(\tau)-\bar{\beta}^{\frac{1}{3}})^2\mathrm{d}\tau\right)^{\frac{1}{2}}\\ =&\bar{\beta}^{\frac{1}{3}}\left(e^{\frac{\sigma^2}{9\theta}}-2e^{\frac{\sigma^2}{36\theta}}+1\right)^{\frac{1}{2}}\ a.s.,\end{aligned}$$

which implies

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{t}\int_0^tF(\beta(\tau))\mathrm{d}\tau=&\frac{c_4c_5\Lambda b}{\mu a}\lim_{t\to+\infty}\frac{1}{t}\int_0^t\left(\beta^2(\tau)-\bar{\beta}^2e^{\frac{\sigma^2}{\theta}}\right)\mathrm{d}\tau\\ &+\frac{3\Lambda b}{\mu a}\bar{\beta}^{\frac{2}{3}}\lim_{t\to+\infty}\frac{1}{t}\int_0^t\left[|\beta^{\frac{1}{3}}(\tau)-\bar{\beta}^{\frac{1}{3}}|-\bar{\beta}^{\frac{1}{3}}\left(e^{\frac{\sigma^2}{9\theta}}-2e^{\frac{\sigma^2}{36\theta}}+1\right)^{\frac{1}{2}}\right]\mathrm{d}\tau\\ \leq&0\ a.s..\end{aligned}\tag{3.16}$$

Because S, I, m are contained in the invariant set Ω, the following equations hold

$$\limsup_{t\to+\infty}\frac{\log S(t)}{t}\leq0,\ \limsup_{t\to+\infty}\frac{\log I(t)}{t}\leq0,\ \limsup_{t\to+\infty}\frac{\log m(t)}{t}\leq0,\ a.s.,\tag{3.17}$$

which means $\lim_{t\to+\infty}\frac{V_3(t)-V_3(0)}{t}=0$. Letting $t\to+\infty$ and combining (3.16) and (3.17), the inequality (3.15) turns into

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^tI(\tau)\mathrm{d}\tau\geq\frac{2ac_5(\mu+\gamma)(R_0^s-1)}{bc_4}\ a.s..$$

If $R_0^s>1$, then $\liminf_{t\to+\infty}\frac{1}{t}\int_0^tI(\tau)\mathrm{d}\tau>0$ $a.s.$. This completes the proof. □

Stationary distribution plays an irreplaceable role in the analysis of disease persistence. Hence, we establish sufficient criterion for the existence of stationary distribution of the stochastic system (1.2) in Theorem 3.3. To proceed, a necessary lemma should be illustrated.

Lemma 3.2. *[31] For any initial value $X(0) = (S(0), I(0), m(0), \beta(0)) \in \Omega$, if there is a bounded closed domain $U \subset \Omega$ with a regular boundary and*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t P(\tau, X(0), U)\mathrm{d}\tau > 0 \ a.s., \tag{3.18}$$

where $P(t, X(0), U)$ is the transition probability of the solution $X(t) = (S(t), I(t), m(t), \beta(t))$, then the stochastic system (1.2) has at least one stationary distribution.

Theorem 3.3. *If $R_0^s > 1$, then the stochastic system (1.2) has a stationary distribution in the invariant set Ω*

Proof. We will prove this theorem in three steps: (i) constructing a non-negative Lyapunov function V; (ii) searching a compact set $U \subset \Omega$ which can ensure that $\mathcal{L}V \le -1$ for any $(S, I, m, \beta) \in \Omega \setminus U$; (iii) verifying that the inequality (3.18) holds.

Step I. (Formulation of the suitable Lyapunov function V)

Define a C^2-Lyapunov function V_4: $\Omega \to \mathbb{R}$ as

$$V_4 = -\log S - \log\left(\frac{b}{a} - m\right) - \log\left(\frac{\Lambda}{\mu} - S - I\right) + \beta - 1 - \log\beta.$$

By Itô's formula and the inequality (2.3), there is

$$\begin{aligned}
\mathcal{L}V_4 =& -\frac{\Lambda}{S} + \frac{\beta m I}{1+hI} + \mu + \frac{m\left(b - am - \frac{\alpha I}{1+hI}\right)}{\frac{b}{a} - m} + \frac{\Lambda - \mu(S+I) - \gamma I}{\frac{\Lambda}{\mu} - S - I} \\
&+ \beta\left(\theta\log\bar{\beta} - \theta\log\beta + \frac{1}{2}\sigma^2\right) - \theta(\log\bar{\beta} - \log\beta) \\
\le& -\frac{\Lambda}{S} + \frac{b}{ah}\beta + \mu + am - \frac{\frac{\alpha m I}{1+hI}}{\frac{b}{a} - m} + \mu - \frac{\gamma I}{\frac{\Lambda}{\mu} - S - I} \\
&+ \beta\left(\theta\log\bar{\beta} - \theta\log\beta + \frac{1}{2}\sigma^2\right) - \theta(\log\bar{\beta} - \log\beta).
\end{aligned} \tag{3.19}$$

Since $\frac{\alpha I}{1+\frac{h\Lambda}{\mu}} \le \frac{\alpha I}{1+hI}$, we can deduce $-\frac{\alpha I}{1+hI} \le -\frac{\alpha\mu I}{h\Lambda+\mu}$. Therefore, we know that

$$\begin{aligned}
\mathcal{L}V_4 \le& -\frac{\Lambda}{S} + \frac{b}{ah}\beta + 2\mu + b - \frac{\alpha\mu m I}{(h\Lambda+\mu)(\frac{b}{a} - m)} - \frac{\gamma I}{\frac{\Lambda}{\mu} - S - I} \\
&+ \beta\left(\theta\log\bar{\beta} - \theta\log\beta + \frac{1}{2}\sigma^2\right) - \theta(\log\bar{\beta} - \log\beta).
\end{aligned} \tag{3.20}$$

Let $V_5 = NV_3 + V_4$, and N is a large enough positive constant satisfying

$$-\frac{1}{2}N(\mu+\gamma)(R_0^s - 1) + \sup_{\beta\in\mathbb{R}_+} g(\beta) \le -2, \tag{3.21}$$

where

$$g(\beta) = 2\mu + b + \frac{b}{ah}\beta + \beta\left(\theta\log\bar{\beta} - \theta\log\beta + \frac{1}{2}\sigma^2\right) - \theta(\log\bar{\beta} - \log\beta).$$

Because the function $V_5 \to +\infty$ as (S, I, m, β) trends to the boundary of the set Ω, there is a minimum value $(V_5)_{\min} \in \Omega$. Finally, we can define a non-negative suitable C^2-Lyapunov function: $\Omega \to \mathbb{R}_+$ as

$$V = V_5 - (V_5)_{\min}.$$

According to (3.13) and (3.20), we have

$$\begin{aligned}
\mathcal{L}V \leq & -\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4}{4c_5}NmI-\frac{\Lambda}{S}-\frac{\alpha\mu m I}{(h\Lambda+\mu)(\frac{b}{a}-m)}-\frac{\gamma I}{\frac{\Lambda}{\mu}-S-I} \\
& +2\mu+b+\frac{b}{ah}\beta+\beta\left(\theta\log\bar{\beta}-\theta\log\beta+\frac{1}{2}\sigma^2\right)-\theta(\log\bar{\beta}-\log\beta)+NF(\beta) \\
= & -\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4}{4c_5}NmI-\frac{\Lambda}{S}-\frac{\alpha\mu m I}{(h\Lambda+\mu)(\frac{b}{a}-m)}-\frac{\gamma I}{\frac{\Lambda}{\mu}-S-I}+g(\beta)+NF(\beta) \\
:= & G(S,I,m,\beta)+NF(\beta).
\end{aligned} \tag{3.22}$$

Step II. (Search of a compact set U)

Now we are in the process to find a compact set $U \subset \Omega$ such that $\mathcal{L}V \leq -1$ for any $(S,I,m,\beta) \in \Omega \setminus U$. Denote

$$U = \left\{(S,I,m,\beta) \in \Omega | S \geq \epsilon,\ I \geq \epsilon,\ \epsilon \leq m \leq \frac{b}{a}-\epsilon^2,\ S+I \leq \frac{\Lambda}{\mu}-\epsilon^2,\ \epsilon \leq \beta \leq \frac{1}{\epsilon}\right\},$$

where ϵ is a positive constant and satisfies the following inequalities

$$-2+\frac{c_4\Lambda b}{4c_5\mu a}N-\min\left\{\frac{\Lambda}{\epsilon},\ \frac{\alpha\mu(\frac{b}{a}-\epsilon^2)}{\epsilon(h\Lambda+\mu)},\ \frac{\gamma}{\epsilon}\right\} \leq -1, \tag{3.23a}$$

$$-2+\max\left\{\frac{c_4 b}{4c_5 a}N\epsilon,\ \frac{c_4\Lambda}{4c_4\mu}N\epsilon\right\} \leq -1, \tag{3.23b}$$

$$-\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4\Lambda b}{4c_5\mu a}N+\frac{1}{2}\theta\log\epsilon+\sup_{\beta\in\mathbb{R}_+}\left[g(\beta)-\frac{1}{2}\theta\log\beta\right] \leq -1, \tag{3.23c}$$

$$-\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4\Lambda b}{4c_5\mu a}N+\frac{\theta\log\epsilon}{2\epsilon}+\sup_{\beta\in\mathbb{R}_+}\left[g(\beta)+\frac{\theta\beta\log\beta}{2}\right] \leq -1. \tag{3.23d}$$

Then the set $\Omega \setminus U$ can be expressed as $\Omega \setminus U = \cup_{i=1}^7 U_i$, and

$$\begin{aligned}
U_1^c = & \{(S,I,m,\beta)\in\Omega | S<\epsilon\},\ U_2^c=\{(S,I,m,\beta)\in\Omega | I<\epsilon\}, U_3^c=\{(S,I,m,\beta)\in\Omega | m<\epsilon\}, \\
U_4^c = & \left\{(S,I,m,\beta)\in\Omega | I\geq\epsilon,\ m>\frac{b}{a}-\epsilon^2\right\},\ U_5^c=\left\{(S,I,m,\beta)\in\Omega | I\geq\epsilon,\ S+I>\frac{\Lambda}{\mu}-\epsilon^2\right\}, \\
U_6^c = & \{(S,I,m,\beta)\in\Omega | \beta<\epsilon\},\ U_7^c=\left\{(S,I,m,\beta)\in\Omega | \beta>\frac{1}{\epsilon}\right\}.
\end{aligned}$$

Case 1. If $(S,I,m,\beta) \in U_1^c$, from (3.21) and (3.23a), we have

$$\begin{aligned}
G(S,I,m,\beta) = & -\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4}{4c_5}NmI-\frac{\Lambda}{S}-\frac{\alpha\mu m I}{(h\Lambda+\mu)(\frac{b}{a}-m)}-\frac{\gamma I}{\frac{\Lambda}{\mu}-S-I}+g(\beta) \\
\leq & -2+\frac{c_4\Lambda b}{4c_5\mu a}N-\frac{\Lambda}{\epsilon} \\
\leq & -2+\frac{c_4\Lambda b}{4c_5\mu a}N-\min\left\{\frac{\Lambda}{\epsilon},\ \frac{\alpha\mu b}{a\epsilon(h\Lambda+\mu)},\ \frac{\gamma}{\epsilon}\right\} \\
\leq & -1.
\end{aligned}$$

Case 2. If $(S,I,m,\beta) \in U_2^c$, from (3.21) and (3.23b), we have

$$\begin{aligned}
G(S,I,m,\beta) = & -\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4}{4c_5}NmI-\frac{\Lambda}{S}-\frac{\alpha\mu m I}{(h\Lambda+\mu)(\frac{b}{a}-m)}-\frac{\gamma I}{\frac{\Lambda}{\mu}-S-I}+g(\beta) \\
\leq & -2+\frac{c_4 b}{4c_5 a}N\epsilon
\end{aligned}$$

$$
\begin{aligned}
&\leq -2+\max\left\{\frac{c_4 b}{4c_5 a}N\epsilon,\ \frac{c_4\Lambda}{4c_4\mu}N\epsilon\right\}\\
&\leq -1.
\end{aligned}
$$

Case 3. If $(S,I,m,\beta)\in U_3^c$, from (3.21) and (3.23b), we have

$$
\begin{aligned}
G(S,I,m,\beta)=&-\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4}{4c_5}NmI-\frac{\Lambda}{S}-\frac{\alpha\mu mI}{(h\Lambda+\mu)(\frac{b}{a}-m)}-\frac{\gamma I}{\frac{\Lambda}{\mu}-S-I}+g(\beta)\\
\leq&-2+\frac{c_4\Lambda}{4c_4\mu}N\epsilon\\
\leq&-2+\max\left\{\frac{c_4 b}{4c_5 a}N\epsilon,\ \frac{c_4\Lambda}{4c_4\mu}N\epsilon\right\}\\
\leq&-1.
\end{aligned}
$$

Case 4. If $(S,I,m,\beta)\in U_4^c$, from (3.21) and (3.23a), we have

$$
\begin{aligned}
G(S,I,m,\beta)=&-\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4}{4c_5}NmI-\frac{\Lambda}{S}-\frac{\alpha\mu mI}{(h\Lambda+\mu)(\frac{b}{a}-m)}-\frac{\gamma I}{\frac{\Lambda}{\mu}-S-I}+g(\beta)\\
\leq&-2+\frac{c_4\Lambda b}{4c_5\mu a}N-\frac{\alpha\mu(\frac{b}{a}-\epsilon^2)}{\epsilon(h\Lambda+\mu)}\\
\leq&-2+\frac{c_4\Lambda b}{4c_5\mu a}N-\min\left\{\frac{\Lambda}{\epsilon},\ \frac{\alpha\mu(\frac{b}{a}-\epsilon^2)}{\epsilon(h\Lambda+\mu)},\ \frac{\gamma}{\epsilon}\right\}\\
\leq&-1.
\end{aligned}
$$

Case 5. If $(S,I,m,\beta)\in U_5^c$, from (3.21) and (3.23a), we have

$$
\begin{aligned}
G(S,I,m,\beta)=&-\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4}{4c_5}NmI-\frac{\Lambda}{S}-\frac{\alpha\mu mI}{(h\Lambda+\mu)(\frac{b}{a}-m)}-\frac{\gamma I}{\frac{\Lambda}{\mu}-S-I}+g(\beta)\\
\leq&-2+\frac{c_4\Lambda b}{4c_5\mu a}N-\frac{\gamma}{\epsilon}\\
\leq&-2+\frac{c_4\Lambda b}{4c_5\mu a}N-\min\left\{\frac{\Lambda}{\epsilon},\ \frac{\alpha\mu(\frac{b}{a}-\epsilon^2)}{\epsilon(h\Lambda+\mu)},\ \frac{\gamma}{\epsilon}\right\}\\
\leq&-1.
\end{aligned}
$$

Case 6. If $(S,I,m,\beta)\in U_6^c$, from and (3.23c), we have

$$
\begin{aligned}
G(S,I,m,\beta)=&-\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4}{4c_5}NmI-\frac{\Lambda}{S}-\frac{\alpha\mu mI}{(h\Lambda+\mu)(\frac{b}{a}-m)}-\frac{\gamma I}{\frac{\Lambda}{\mu}-S-I}+g(\beta)\\
\leq&-\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4\Lambda b}{4c_5\mu a}N+\frac{1}{2}\theta\log\epsilon+\sup_{\beta\in\mathbb{R}_+}\left[g(\beta)-\frac{1}{2}\theta\log\beta\right]\\
\leq&-1.
\end{aligned}
$$

Case 7. If $(S,I,m,\beta)\in U_7^c$, from (3.23d), we have

$$
\begin{aligned}
G(S,I,m,\beta)=&-\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4}{4c_5}NmI-\frac{\Lambda}{S}-\frac{\alpha\mu mI}{(h\Lambda+\mu)(\frac{b}{a}-m)}-\frac{\gamma I}{\frac{\Lambda}{\mu}-S-I}+g(\beta)\\
\leq&-\frac{1}{2}N(\mu+\gamma)(R_0^s-1)+\frac{c_4\Lambda b}{4c_5\mu a}N+\frac{\theta\log\epsilon}{2\epsilon}+\sup_{\beta\in\mathbb{R}_+}\left[g(\beta)+\frac{\theta\beta\log\beta}{2}\right]\\
\leq&-1.
\end{aligned}
$$

By the **Case 1-7**, we get the conclusion that

$$G(S,I,m,\beta) \leq -1,\ \forall (S,I,m,\beta) \in \Omega \setminus U. \tag{3.24}$$

Until now, the compact set $U \subset \Omega$ is established.

Step III. Verify of the inequality (3.18)

On the other hand, we can find a positive constant H to make sure that

$$G(S,I,m,\beta) \leq H,\ \forall (S,I,m,\beta) \in \Omega. \tag{3.25}$$

Integrating (3.22) over the interval $[0,t]$ and taking expectations obtain

$$\begin{aligned}
0 \leq & \mathbb{E}\left[V\left(S(t),I(t),m(t),\beta(t)\right)\right] \\
= & \mathbb{E}\left[V\left(S(0),I(0),m(0)\beta(0)\right)\right] + \int_0^t \mathbb{E}\left[\left(G\left(S(\tau),I(\tau),m(\tau),\beta(\tau)\right)\right)\right] \mathrm{d}\tau + N\int_0^t \mathbb{E}(F(\beta(\tau)))\mathrm{d}\tau.
\end{aligned} \tag{3.26}$$

From (3.16), dividing t on the both sides of (3.26) and letting $t \to +\infty$ follow

$$\begin{aligned}
0 \leq & \liminf_{t\to+\infty} \frac{1}{t}\int_0^t \mathbb{E}\left[V\left(S(t),I(t),m(t),\beta(t)\right)\right] \mathrm{d}\tau \\
= & \liminf_{t\to+\infty} \frac{1}{t}\int_0^t \mathbb{E}\left[\left(G\left(S(\tau),I(\tau),m(\tau)\beta(\tau)\right)\right)\right] \mathrm{d}\tau + N\liminf_{t\to+\infty} \frac{1}{t}\int_0^t \mathbb{E}(F(\beta(\tau)))\mathrm{d}\tau \\
\leq & \liminf_{t\to+\infty} \frac{1}{t}\int_0^t \mathbb{E}\left[\left(G\left(S(\tau),I(\tau),m(\tau)\beta(\tau)\right)\right)\right] \mathrm{d}\tau \ a.s..
\end{aligned} \tag{3.27}$$

Combining (3.24) and (3.25), the inequality (3.27) can be expressed as

$$\begin{aligned}
0 \leq & \liminf_{t\to+\infty} \frac{1}{t}\int_0^t \mathbb{E}\left[V\left(S(t),I(t),m(t),\beta(t)\right)\right] \mathrm{d}\tau \\
\leq & \liminf_{t\to+\infty} \frac{1}{t}\int_0^t \mathbb{E}\left[G\left(S(\tau),I(\tau),m(\tau),\beta(\tau)\right)\mathbf{1}_{\{(S(\tau),I(\tau),m(\tau),\beta(\tau))\in U\}}\right] \mathrm{d}\tau \\
& + \liminf_{t\to+\infty} \frac{1}{t}\int_0^t \mathbb{E}\left[G\left(S(\tau),I(\tau),m(\tau),\beta(\tau)\right)\mathbf{1}_{\{(S(\tau),I(\tau),m(\tau),\beta(\tau))\in \Omega\setminus U\}}\right] \mathrm{d}\tau \\
\leq & H\liminf_{t\to+\infty} \frac{1}{t}\int_0^t P\left(\left(S(\tau),I(\tau),m(\tau),\beta(\tau)\right)\in U\right)\mathrm{d}\tau - \liminf_{t\to+\infty} \frac{1}{t}\int_0^t P\left(\left(S(\tau),I(\tau),m(\tau),\beta\right)\in \Omega\setminus U\right)\mathrm{d}\tau \\
\leq & -1 + (H+1)\liminf_{t\to+\infty} \frac{1}{t}\int_0^t P\left(\left(S(\tau),I(\tau),m(\tau),\beta(\tau)\right)\in U\right)\mathrm{d}\tau \ a.s.,
\end{aligned} \tag{3.28}$$

where $\mathbf{1}_{\{X\}}$ denotes the indicator function. This means that, for any initial value $(S(0),I(0),m(0),\beta(0)) \in \Omega$, the following inequality holds

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t P(\tau,(S(0),I(0),m(0),\beta(0)),U)\mathrm{d}\tau \geq \frac{1}{H+1} > 0 \ a.s..$$

Therefore, the stochastic system (1.2) admits at least one stationary distribution in the invariant set Ω by Lemma 3.2, which implies that the influenza will prevail. The proof ends. □

Remark 3.1. *From the expression of R_0, R_0^e and R_0^s, we get $R_0^s \leq R_0 \leq R_0^e$. For one thing, it is obvious that $R_0 > 1$ if $R_0^s > 1$, and $R_0 < 1$ when $R_0^e < 1$, this means $R_0^s > 1$ or $R_0^e < 1$ can also be the sufficient conditions for the influenza persistence or extinction of the deterministic system (1.1). For another thing, $R_0^s = R_0 = R_0^e$ holds if and only if the intensity of environmental perturbations $\theta = 0$, which implies that the dynamical properties of the stochastic system (1.2) are consistent with the deterministic model (1.1) when there are no environmental perturbations.*

4. Probability density function

Theorem 3.3 shows that the stochastic system (1.2) has a stationary distribution when $R_0^s > 1$. Since the probability function can greatly reveal the dynamical and statistical properties, it is more specific and comprehensive than the stationary distribution. In this section, we will give the explicit expression of the probability function of stationary distribution near the quasi-endemic equilibrium. The process is divided into two steps: (i) linearizing the stochastic system (1.2) around the quasi-endemic equilibrium; (ii) solving the corresponding four-dimensional Fokker-Planck equation.

4.1. Linearization of the system (1.2)

To get the corresponding linearized system of (1.2), we first have to define the quasi-endemic equilibrium E^* involved in stochasticity, which is calculated by

$$\begin{cases} \Lambda - \mu S^* - \frac{\beta^* m^* I^* S^*}{1+hI^*} = 0, \\ \frac{\beta^* m^* I^* S^*}{1+hI^*} - (\mu+\gamma) I^* = 0, \\ bm^* - a(m^*)^2 - \frac{\alpha I^* m^*}{1+hI^*} = 0, \\ \log \bar{\beta} - \log \beta^* = 0. \end{cases}$$

From the former discussion, $E^* = \left(S^*, I^*, m^*, \log \bar{\beta}\right)$ exists when $R_0 > 1$. Let $Y^T = (y_1, y_2, y_3, y_4)^T = \left(S - S^*, I - I^*, m - m^*, \log \beta - \log \bar{\beta}\right)^T$, then the stochastic system (1.2) can be linearized around E^* as follows

$$\begin{cases} \mathrm{d}y_1 = (-a_{11}y_1 - a_{12}y_2 - a_{13}y_3 - a_{14}y_4)\mathrm{d}t, \\ \mathrm{d}y_2 = (a_{21}y_1 - a_{22}y_2 + a_{13}y_3 + a_{14}y_4)\mathrm{d}t, \\ \mathrm{d}y_3 = (-a_{32}y_2 - a_{33}y_3)\mathrm{d}t, \\ \mathrm{d}y_4 = -a_{44}y_4\mathrm{d}t + \sigma \mathrm{d}B(t), \end{cases} \tag{4.1}$$

where

$$\begin{aligned} a_{11} =& \frac{\bar{\beta} m^* I^*}{1+hI^*} + \mu, \ a_{12} = \frac{\bar{\beta} m^* S^*}{1+hI^*} - \frac{\bar{\beta} h m^* S^* I^*}{(1+hI^*)^2}, \ a_{13} = \frac{\bar{\beta} S^* I^*}{1+hI^*}, \ a_{14} = \frac{\bar{\beta} m^* S^* I^*}{1+hI^*}, \ a_{21} = \frac{\bar{\beta} m^* I^*}{1+hI^*}, \\ a_{22} =& \frac{\bar{\beta} h m^* S^* I^*}{(1+hI^*)^2}, \ a_{32} = m^* \left(\frac{\alpha}{1+hI^*} - \frac{\alpha h I^*}{(1+hI^*)^2} \right), \ a_{33} = am^*, \ a_{44} = \theta. \ a_{ij} > 0 \ (i,j = 1,2,3,4). \end{aligned}$$

4.2. Calculation of the corresponding Fokker-Planck equation

Before solving the matrix equation to get the expression of the probability function, we give the following needed definition and one lemma.

Definition 4.1. *[32] Let $\phi_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$ be the characteristic polynomial of n-dimensional square matrix A, then A is a Hurwitz matrix if and only its eigenvalues all have negative real-part. That is equal to*

$$H_k = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ 1 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ 0 & 1 & a_2 & \cdots & a_{2k-4} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_k \end{vmatrix} > 0,$$

$k = 1, 2, \cdots, n$, among them $j > n$, replenishing definition $a_j = 0$.

Lemma 4.1. *[33] For the four-dimensional algebraic equation $G_0^2 + K\Theta + \Theta K^T = 0$, where $G_0 = diag(1,0,0,0)$ and*

$$K = \begin{pmatrix} -k_1 & -k_2 & -k_3 & -k_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

the solution Θ takes the form

$$\Theta = \begin{pmatrix} \theta_{11} & 0 & \theta_{13} & 0 \\ 0 & -\theta_{13} & 0 & \theta_{24} \\ \theta_{13} & 0 & -\theta_{24} & 0 \\ 0 & \theta_{24} & 0 & \theta_{44} \end{pmatrix},$$

in which

$$\begin{aligned} k =& 2\left[k_1(k_1k_4 - k_2k_3) + k_3^2\right], \\ \theta_{11} =& \frac{k_1k_4 - k_2k_3}{k}, \ \theta_{13} = \frac{k_3}{k}, \ \theta_{24} = \frac{k_1}{k}, \ \theta_{44} = \frac{k_3 - k_1k_2}{k_4k}. \end{aligned}$$

What is more, Θ is a symmetric and positive definite matrix with conditions $k_1 > 0$, $k_3 > 0$, $k_4 > 0$ and $k_1(k_2k_3 - k_1k_4) - k_3^2 > 0$.

Theorem 4.1. *Suppose that $R_0^s > 1$ and $\omega = \frac{a_{32}(\mu - am^*)}{\gamma} \neq 0$, then the distribution of the solution $(S, I, m, \log\beta)$ of stochastic system (1.2) follows the normal density function $\Phi(S, I, m, \log\beta)$ near the quasi-endemic equilibrium E^*, which has the following expression*

$$\Phi(S, I, m, \log\beta) = (2\pi)^{-2}|\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(S-S^*, I-I^*, m-m^*, \log\beta - \log\bar{\beta})\Sigma^{-1}(S-S^*, I-I^*, m-m^*, \log\beta-\log\bar{\beta})^T}.$$

The expression of covariance matrix Σ will be confirmed later.

Proof. For the sake of simplicity, let $\mathbf{B}(t) = (0, 0, 0, B(t))^T$,

$$A = \begin{pmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ a_{21} & -a_{22} & a_{13} & a_{14} \\ 0 & -a_{32} & -a_{33} & 0 \\ 0 & 0 & 0 & -a_{44} \end{pmatrix}, \ G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma \end{pmatrix}.$$

With those symbols, the system (4.1) can be expressed as

$$\mathrm{d}Y(t) = AY\mathrm{d}t + G\mathrm{d}\mathbf{B}(t). \tag{4.2}$$

Moreover, we can rewrite the matrix A as

$$A = \begin{pmatrix} B_{3\times 3} & C_{3\times 1} \\ \mathbf{0}_{1\times 3} & -a_{44} \end{pmatrix}.$$

From conclusions in [15], the eigenvalues of matrix $B_{3\times 3}$ all have negative real parts, meaning that the real part of eigenvalues of matrix A are also all negative, so A is a Hurwitz matrix. In the view of Gardiner's theory [34], the equation (4.2) admits a unique probability density $\Phi(S, I, m, \log\beta)$, and it is determined by the following four-dimensional Fokker-Planck equation

$$\frac{\partial}{\partial t}\Phi(Y(t), t) + \frac{\partial}{\partial Y(t)}[AY(t)\Phi(Y(t), t)] - \frac{\sigma^2}{2}\frac{\partial^2}{\partial y_4^2}\Phi(Y(t), t) = 0. \tag{4.3}$$

Since $\frac{\partial}{\partial t}\Phi(Y(t), t) = 0$ under a stationary distribution, the equation (4.3) becomes

$$\begin{aligned} &\frac{\partial}{\partial y_1}[(-a_{11}y_1 - a_{12}y_2 - a_{13}y_3 - a_{14}y_4)\Phi] + \frac{\partial}{\partial y_2}[(a_{21}y_1 - a_{22}y_2 + a_{13}y_3 + a_{14}y_4)\Phi] \\ &+ \frac{\partial}{\partial y_3}[(-a_{32}y_2 - a_{33}y_3)\Phi] - \frac{\sigma^2}{2}\frac{\partial^2}{\partial y_4^2}\Phi = 0. \end{aligned} \tag{4.4}$$

Because the diffusion matrix G is a constant matrix, then the probability density function Φ follows a Gaussian distribution by the theory of Roozen [35], and the covariance matrix Σ is calculated by

$$G^2 + A\Sigma + \Sigma A^T = 0. \tag{4.5}$$

Next, we are devoted to solving equation (4.5) to get the expression of probability function of stationary distribution near E^*.

Let $A_1 = J_1 A J_1^{-1}$, where

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

then we obtain

$$A_1 = \begin{pmatrix} -a_{44} & 0 & 0 & 0 \\ -a_{14} & a_{12} - a_{11} & -a_{12} & -a_{13} \\ 0 & \gamma & -a_{12} - a_{22} & 0 \\ 0 & a_{32} & -a_{32} & -a_{33} \end{pmatrix}.$$

Denote

$$J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{a_{32}}{\gamma} & 1 \end{pmatrix},$$

we calculate $A_2 = J_2 A_1 J_2^{-1}$ and get

$$A_2 = \begin{pmatrix} -a_{44} & 0 & 0 & 0 \\ -a_{14} & a_{12} - a_{11} & -\frac{a_{13}a_{32}}{\gamma} - a_{12} & -a_{13} \\ 0 & \gamma & -a_{12} - a_{22} & 0 \\ 0 & 0 & \omega & -a_{33} \end{pmatrix}.$$

Motivated by the method in [33], the transmission matrix of A_2 is given by

$$T = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ 0 & \gamma\omega & -\omega(a_{12} + a_{22} + a_{33}) & a_{33}^2 \\ 0 & 0 & \omega & -a_{33} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

in which

$$\begin{aligned} t_1 =& - \gamma\omega a_{14},\ t_2 = -\gamma\omega(a_{11} + a_{22} + a_{33}), \\ t_3 =& \omega \left[(a_{12} + a_{22})(a_{12} + a_{22} + a_{33}) + a_{33}^2 - (a_{13}a_{32} + \gamma a_{12}) \right],\ t_4 = -(\gamma\omega a_{13} + a_{33}^3). \end{aligned}$$

Define $A_3 = T A_2 T^{-1}$, by a simple calculation, we have

$$A_3 = \begin{pmatrix} -r_1 & -r_2 & -r_3 & -r_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where

$$\begin{aligned} r_1 =& a_{11} + a_{22} + a_{33} + a_{44} \triangleq p_1 + a_{44}, \\ r_2 =& a_{11}a_{22} + a_{12}a_{21} + a_{11}a_{33} + a_{13}a_{32} + a_{22}a_{33} + p_1 a_{44} \triangleq p_2 + p_1 a_{44}, \\ r_3 =& a_{11}a_{22}a_{33} + a_{11}a_{13}a_{32} + a_{12}a_{21}a_{33} - a_{13}a_{21}a_{32} + p_2 a_{44} \triangleq p_3 + p_2 a_{44}, \\ r_4 =& p_3 a_{44}. \end{aligned}$$

Therefore, the equation (4.5) can be transformed into

$$(TJ_2J_1)G^2(TJ_2J_1)^T + A_3(TJ_2J_1)\Sigma(TJ_2J_1)^T + (TJ_2J_1)\Sigma(TJ_2J_1)^T A_3^T = 0. \tag{4.6}$$

Let $\rho = t_1\sigma$ and $\tilde{\Sigma} = \rho^{-2}(TJ_2J_1)\Sigma(TJ_2J_1)^T$, then equation (4.6) is equal to

$$G_0^2 + A_3\tilde{\Sigma} + \tilde{\Sigma}A_3^T = 0.$$

According to the Lemma 4.1, we have

$$\tilde{\Sigma} = \begin{pmatrix} \sigma_{11} & 0 & \sigma_{13} & 0 \\ 0 & -\sigma_{13} & 0 & \sigma_{24} \\ \sigma_{13} & 0 & -\sigma_{24} & 0 \\ 0 & \sigma_{24} & 0 & \sigma_{44} \end{pmatrix},$$

where

$$\begin{aligned} r =& 2\left[r_1(r_1r_4 - r_2r_3) + r_3^2\right], \\ \sigma_{11} =& \frac{r_1r_4 - r_2r_3}{r},\ \sigma_{13} = \frac{r_3}{r},\ \sigma_{24} = \frac{r_1}{r},\ \sigma_{44} = \frac{r_3 - r_1r_2}{r_4r}. \end{aligned}$$

From the analysis in [15], we know that $p_1 > 0,\ p_2 > 0,\ p_3 > 0$ and $p_1p_2 - p_3 > 0$, which can deduce $r_i > 0\ (i = 1, 2, 3, 4)$ and

$$\begin{aligned} r_1(r_2r_3 - r_1r_4) - r_3^2 =& (p_1 + a_{44})\left[(p_2 + p_1a_{44})(p_3 + p_2a_{44}) - p_3a_{44}(p_1 + a_{44})\right] - (p_3 + p_2a_{44})^2 \\ =& (p_1p_2 - p_3)a_{44}^3 + p_1(p_1p_2 - p_3)a_{44}^2 + p_2(p_1p_2 - p_3)a_{44} + p_3(p_1p_2 - p_3) \\ >& 0, \end{aligned}$$

This implies $\tilde{\Sigma}$ is a positive definite matrix. Finally, we get the expression of the covariance matrix $\Sigma = \rho^2(TJ_2J_1)^{-1}\tilde{\Sigma}[(TJ_2J_1)^{-1}]^T$ and Σ is also a positive definite matrix. The proof completes. □

5. Numerical simulations

In this section, several numerical simulations are performed to illustrate obtained analytical results. We mainly focus on the four aspects: (i) the local asymptotic stability of endemic equilibrium E_2 in the deterministic model (1.1); (ii) the correctness of Theorem 3.1, 3.3 and 4.1; (iii) the impact of the social capacity of population mobility $\frac{b}{a}$ and transmission rate $\bar{\beta}$ on the disease spread; (iv) the influence of environmental perturbations on the disease extinction and persistence.

According to the higher-order numerical method developed by Milstein [36], the corresponding discretization equation of stochastic system (1.2) takes the form as

$$\begin{cases} S^{j+1} = S^j + \left[\Lambda - \mu S^j - \frac{\bar{\beta}e^{x_j}S^jI^jm^j}{1+hJ^j}\right]\Delta t, \\ I^{j+1} = I^j + \left[\frac{\bar{\beta}e^{x_j}S^jI^jm^j}{1+hJ^j} - (\mu+\gamma)I^j\right]\Delta t, \\ m^{j+1} = m^j + \left[m^j\left(b - am^j - \frac{\alpha I^j}{1+hI^j}\right)\right]\Delta t, \\ x^{j+1} = x^j - \theta x^j\Delta t + \sigma\sqrt{\Delta t}\xi_j + \frac{\sigma^2}{2}(\xi_j^2 - 1)\Delta t, \end{cases}$$

where the value of the jth iteration of (S, I, m, x) is depicted by (S^j, I^j, m^j, x^j), Δt is the time increment and $\xi_j\ (j = 1, 2, \cdots, n)$ represent the mutually independent Gaussian random variables satisfying the standard normal distribution $N(0, 1)$.

Example 5.1 Let

$$\Pi = 0.5,\ \mu = 0.2,\ \bar{\beta} = 0.6,\ h = 3,\ \gamma = 0.15,\ b = 3,\ a = 1,\ \alpha = 1, \tag{5.1}$$

then simple calculations lead to $R_0 = 12.8571 > 1$ and $E_2 = (0.8216, 0.9591, 2.7526)$, which implies that E_2 is locally asymptotically stable. We select five different $(S(0), I(0), m(0))$, and fig.1 represents the trajectories of model (1.1) with initial values $(1.5, 0.5, 2.5)$, $(1.2, 0.3, 1.5)$, $(0.6, 0.6, 0.5)$, $(1, 0.8, 2)$, $(0.9, 07, 1)$, respectively. It is obvious that E_2 is locally asymptotically stable when the basic reproduction number $R_0 > 1$.

Example 5.2 We set $\Pi = 0.1,\ b = 1,\ \theta = 0.5,\ \sigma = 0.4$ and the rest parameters are the same as in (5.1), then it follows that $R_0 = 0.8571$, $R_0^e = 0.9285 < 1$ and $E_1 = (0.5, 0, 1)$, which satisfies Theorem 3.1. The fig.2

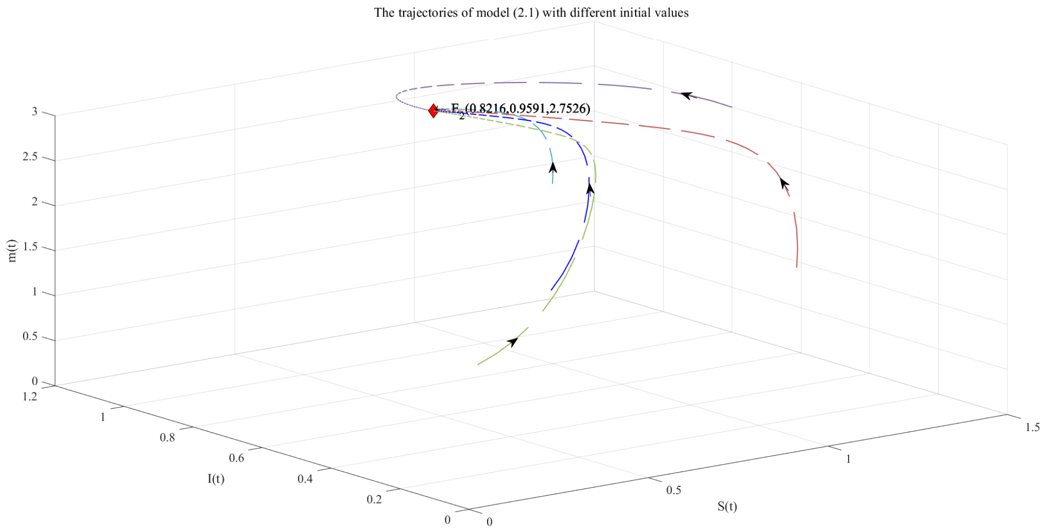

fig. 1: The trajectories of model (1.1) with different initial values $(1.5, 0.5, 2.5)$, $(1.2, 0.3, 1.5)$, $(0.6, 0.6, 0.5)$, $(1, 0.8, 2)$, $(0.9, 07, 1)$, respectively.

shows the variation of $S(t)$, $I(t)$ and $m(t)$ of the deterministic model (1.1) and stochastic model (1.2), implying that the disease dies out.

Example 5.3 Assume that $\theta = 0.5$, $\sigma = 0.03$ and other parameters satisfy the equations (5.1), then there are $R_0^s = 12.4714$ and the quasi-endemic equilibrium $E^* = (0.8216, 0.9591, 2.7526, \log 0.6)$. The covariance matrix Σ can be expressed as

$$\Sigma = 1 \times 10^{-3} \begin{pmatrix} 0.1778 & -0.1297 & 0.0086 & -0.2905 \\ -0.1297 & 0.1022 & -0.0065 & 0.2393 \\ 0.0086 & -0.0065 & 0.0004 & -0.0135 \\ -0.2905 & 0.2393 & -0.0135 & 0.9000 \end{pmatrix},$$

which can deduce the following three marginal density functions

$$\begin{aligned} \frac{\partial \Phi}{\partial S} =& 29.85 e^{-5556(S-0.8216)^2}, \\ \frac{\partial \Phi}{\partial I} =& 40 e^{-10000(I-0.9591)^2}, \\ \frac{\partial \Phi}{\partial m} =& 634.92 e^{-2500000(m-2.7526)^2}. \end{aligned}$$

Therefore, the probability density function of the stationary distribution near E^* is coincident with the function

$$\Phi(S, I, m, \log \beta) = (2\pi)^{-2} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(S-S^*, I-I^*, m-m^*, \log \beta - \log \bar{\beta}) \Sigma^{-1} (S-S^*, I-I^*, m-m^*, \log \beta - \log \bar{\beta})^T}.$$

The fig.3 realizes the variation of $S(t)$, $I(t)$ and $m(t)$ of the deterministic model (1.1) and stochastic model (1.2), and it is clear that the values are stable around the quasi-endemic equilibrium E^*. What is more, fig.4 gives the frequency histograms and marginal density function curves of $S(t)$, $I(t)$ and $m(t)$ of the stochastic model (1.2) in the left column, and the frequency fitting density functions and marginal density functions are represented in the right column. From those two pictures, we can see that influenza will eventually prevail, which can also strongly support Theorem 3.3 and 4.1.

Example 5.4 Population mobility is inevitable in modern society with convenient and fast means of transportation, but if influenza is harder to control and makes huge damage to human health, then governmental agencies may take the implementation of lockdowns to maintain social distance to reduce the loss caused by influenza. This will influence the social capacity of population mobility $\frac{b}{a}$. Also, with the increasing number

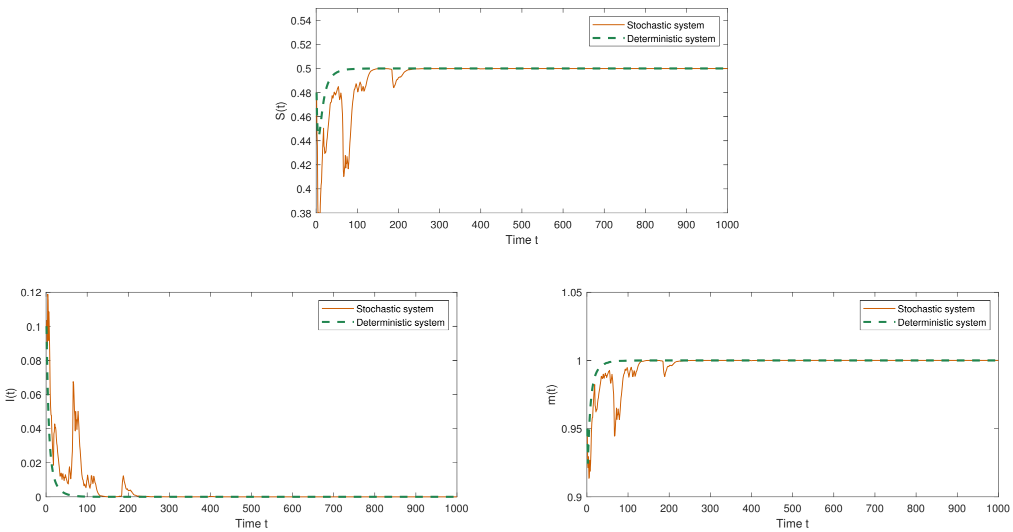

fig. 2: The variation of $S(t)$, $I(t)$ and $m(t)$ of the deterministic model (1.1) and stochastic model (1.2) under example 5.2.

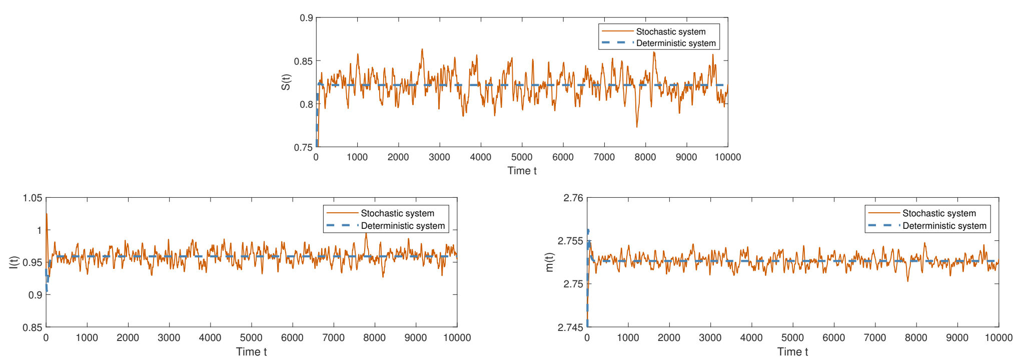

fig. 3: The variation of $S(t)$, $I(t)$ and $m(t)$ of the deterministic model (1.1) and stochastic model (1.2) under example 5.3.

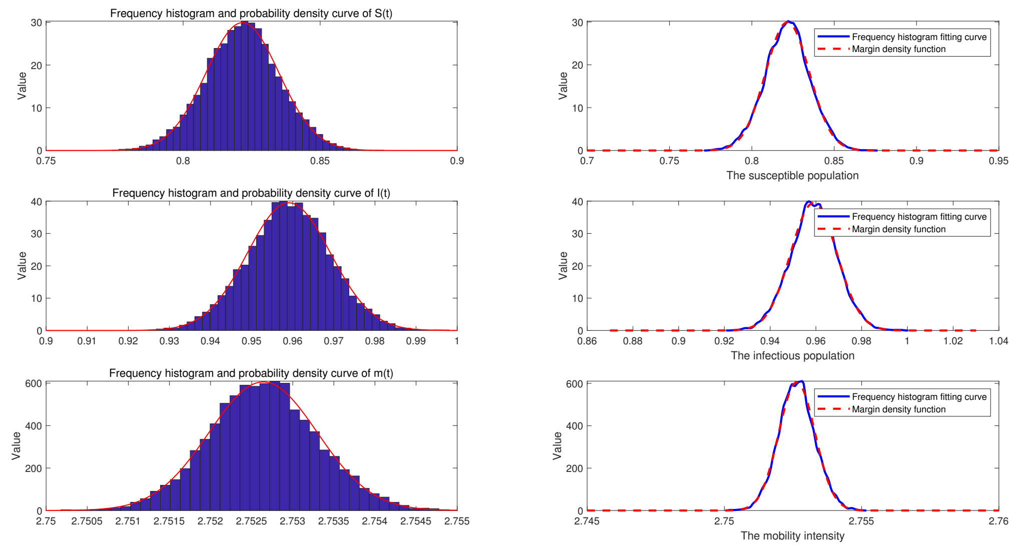

fig. 4: Left column are frequency histograms and marginal density function curves of $S(t)$, $I(t)$ and $m(t)$ in the stochastic model (1.2), respectively. Right column represent frequency fitting density functions and marginal density functions of $S(t)$, $I(t)$ and $m(t)$, respectively.

of infectious population, people would take proper precautionary measures for disease transmission (i.e., use of face mask, social distancing and implementing hygiene), this has an effect on the transmission rate $\bar{\beta}$. We use the same parameters in example 5.3. On the one hand, fig.5 and 7 show the variation trend of R_0, R_0^e and R_0^s with the social capacity of population mobility $\frac{b}{a} \in [0, 1]$ and the transmission rate $\bar{\beta} \in [0, 0.15]$, respectively. On the other hand, fig.6 and fig.8 give the trajectories of $S(t)$, $I(t)$ and $m(t)$ with $\frac{b}{a} = 0.3,\ 0.5,\ 0.8$ and $\bar{\beta} = 0.1,\ 0.3,\ 0.5$, respectively. Those all indicate that reducing population mobility or autonomously taking necessary protection measures can help control influenza. However, the excessive lockdown will hinder economic development. The issue of how to balance the intensity of blockade and development is discussed in [37].

Example 5.5 For one thing, let the parameters be the same as in example 5.2 but $\theta \in [0.1, 1]$ and $\sigma \in [0.1, 1]$. We carry out fig.9 to show the value of R_0^e with different reversion speed and volatility intensity. Obviously, R_0^e decreases with the increase of θ and the decrease of σ, which means that influenza inclines to die out. For another, let $\sigma = 0.03$ and other parameters satisfy the equations (5.1), then fig.10 demonstrates the trajectories of $S(t)$, $I(t)$ and $m(t)$ with different reversion speeds $\theta = 0.2,\ 0.5,\ 1$, respectively. Suppose that $\theta = 0.5$, fig.11 exhibits the trajectories under different volatility intensities $\sigma = 0.02,\ 0.05,\ 0.07$, respectively. Those two pictures reveal that a bigger reversion speed or smaller volatility intensity makes the population more stable.

6. Conclusion and discussions

In this paper, we construct a stochastic epidemic model driven by the log-normal Ornstein-Uhlenbeck process to investigate the dynamical behaviors of influenza transmission. Based on the present research about the mean-reverting Ornstein-Uhlenbeck process, we find that using the mean-reverting process rather than the linear functions of white noise to simulate the environmental interference is more realistic in biological significance. However, up to now, there are few achievements in the basic theories of the mean-reverting process. Hence, in this article, we concentrate on the methods used in the investigation of dynamic properties of the stochastic model governed by the OU process. To be specific, the dynamic behaviors of the influenza system (1.2) can be summarized by the following conclusions:

• The stochastic model admits a unique global positive solution. Furthermore, if the initial value satisfies a certain condition, then the solution will remain in an invariant set a.s.. This is the basis for the next research.

• By constructing suitable Lyapunov functions, two critical conditions

$$R_0^e = R_0 e^{\frac{\sigma^2}{4\theta}} < 1,\ R_0^s = R_0 \left[1 - 3\left(e^{\frac{\sigma^2}{9\theta}} - 2e^{\frac{\sigma^2}{36\theta}} + 1\right)^{\frac{1}{2}}\right] > 1$$

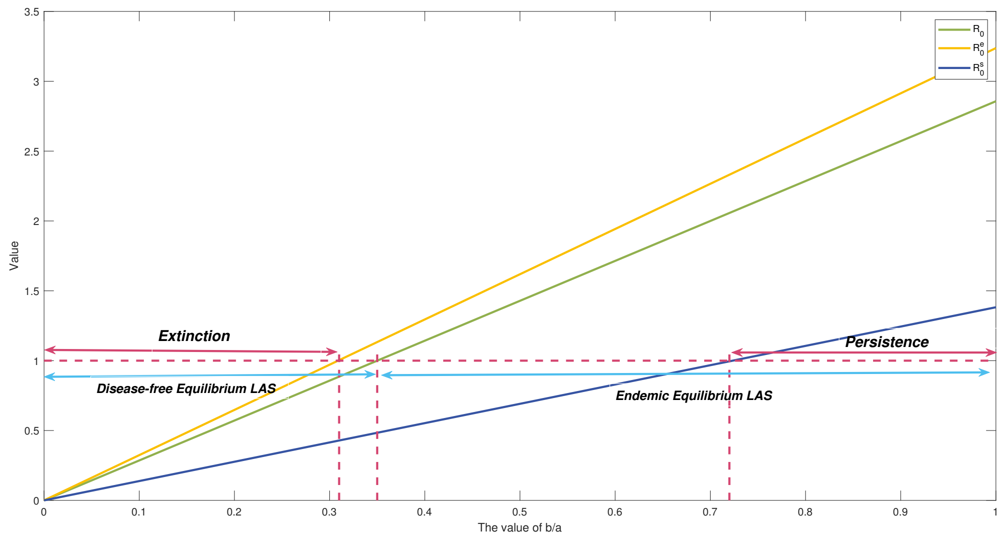

fig. 5: The variation trend of R_0, R_0^e and R_0^s with the social capacity of population mobility $\frac{b}{a} \in [0, 1]$, respectively.

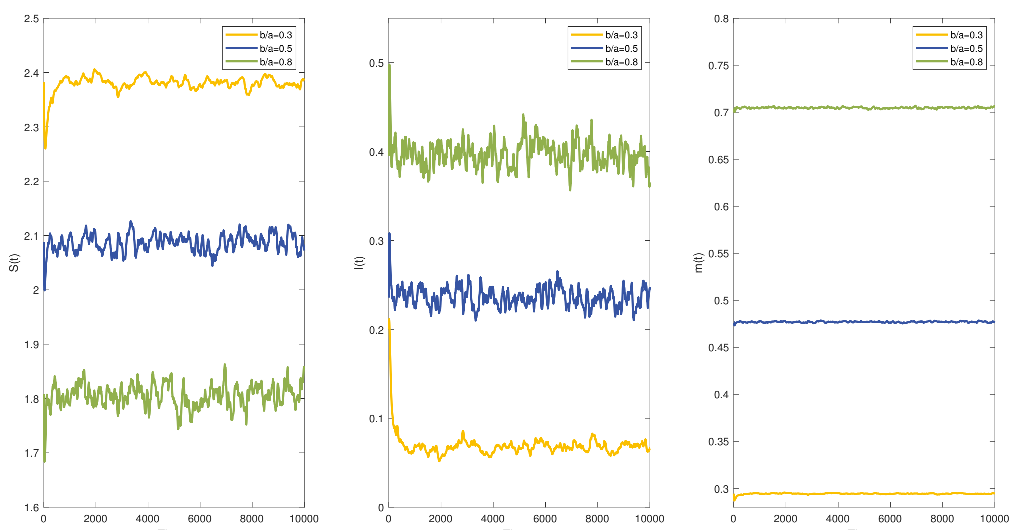

fig. 6: The trajectories of $S(t)$, $I(t)$ and $m(t)$ with $\frac{b}{a} = 0.3,\ 0.5,\ 0.8$, respectively.

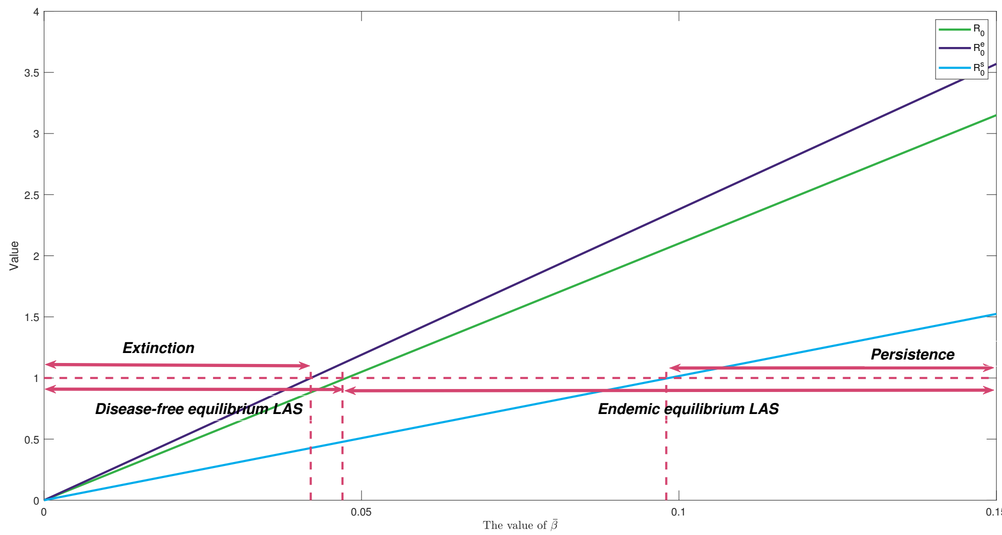

fig. 7: The variation trend of R_0, R_0^e and R_0^s with the transmission rate $\bar{\beta} \in [0, 0.15]$, respectively.

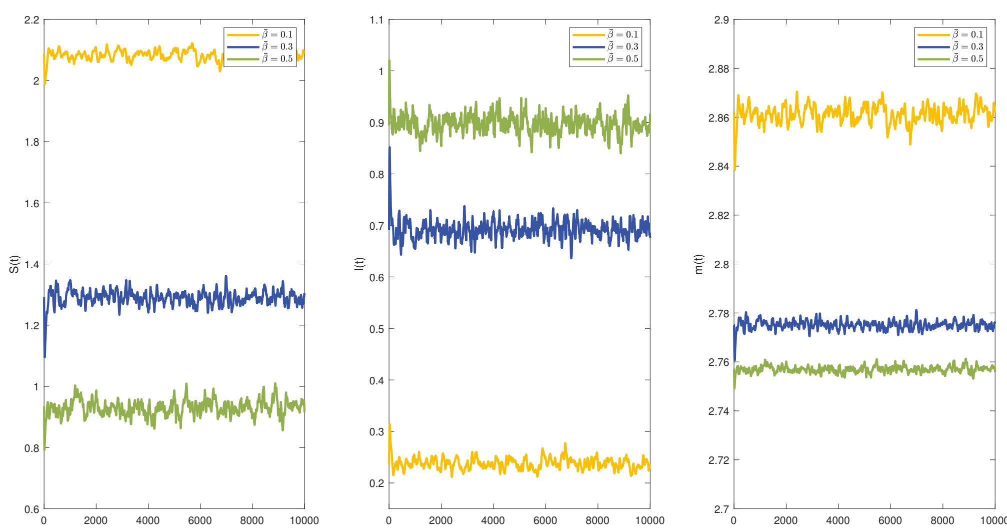

fig. 8: The trajectories of $S(t)$, $I(t)$ and $m(t)$ with $\bar{\beta} = 0.1,\ 0.3,\ 0.5$, respectively.

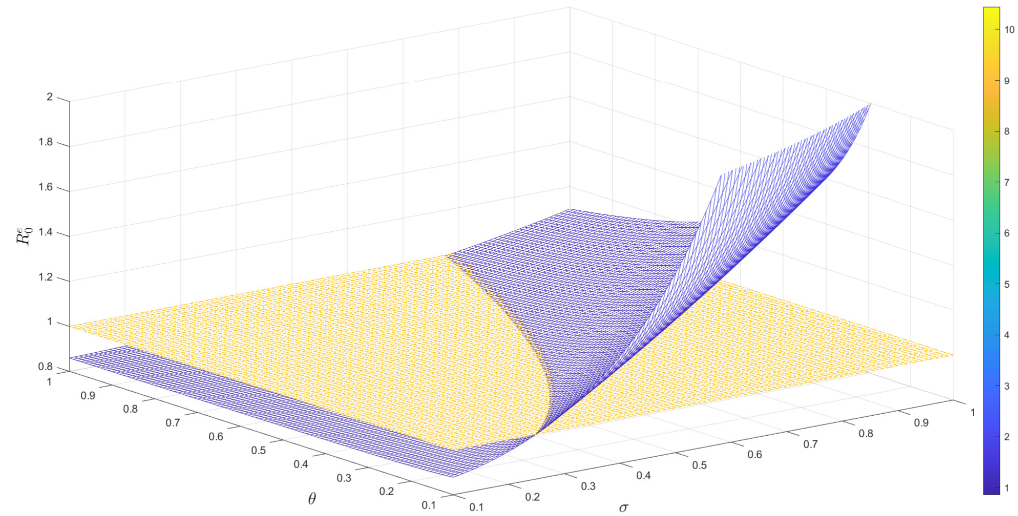

fig. 9: The plane $R_0^e = 1$ and the three-dimensional diagram of R_0^e with $\theta \in [0.1, 1]$ and $\sigma \in [0.1, 1]$.

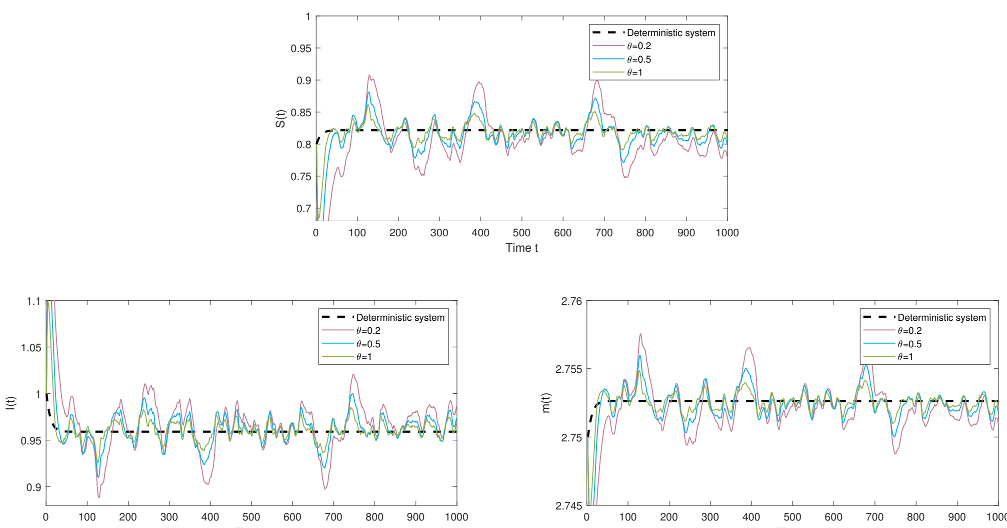

fig. 10: The trajectories of $S(t)$, $I(t)$ and $m(t)$ with volatility intensity $\sigma = 0.03$ and different reversion speed $\theta = 0.2,\ 0.5,\ 1$, respectively.

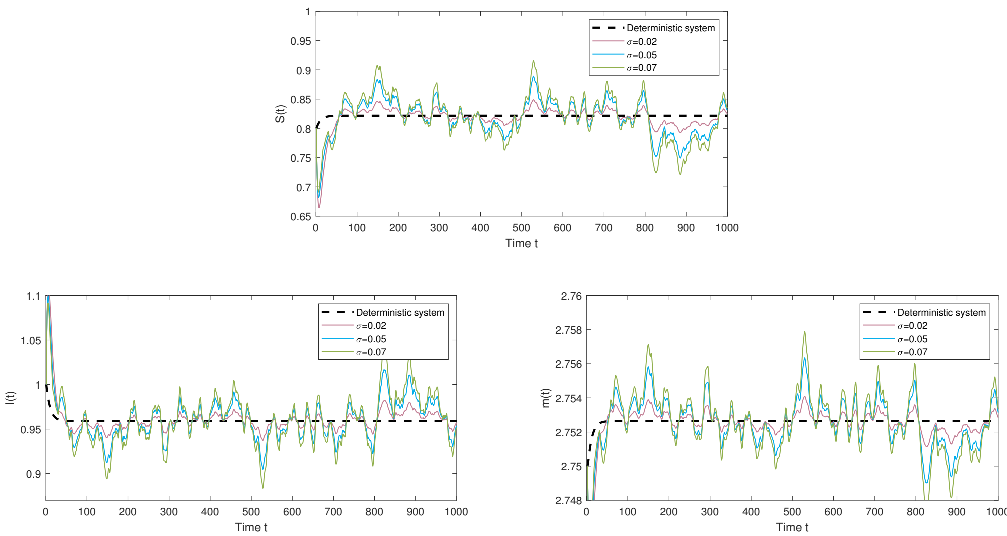

fig. 11: The trajectories of $S(t)$, $I(t)$ and $m(t)$ with reversion speed $\theta = 0.5$ and different volatility intensity $\sigma = 0.02,\ 0.05,\ 0.07$, respectively.

are established for the influenza exponential extinction and persistence. It is worth noting that $R_0^e \to R_0$ and $R_0^s \to R_0$ as the volatility intensity $\sigma \to 0$, in which R_0 is the basic reproduction number in the corresponding deterministic model.

• Based on the theories developed by Gardiner and Roozen, we obtain the probability density function of stationary distribution, which can reveal lots of statistical properties.

In the end, we propose several issues that hope to be better solved in future studies. On the one hand, the social capacity of population mobility in the context of economics has important impacts on disease transmission. Globalization and rapid changes in transportation have made it easier for people to widely travel. Therefore, influenza viruses can spread around the world as people move, putting enormous pressure on disease control. However, a strict blockade will set barriers to economic development and life. It is necessary to select a suitable b/a to minimize the damage caused by influenza and ensure the normal operation of society. On the other hand, limited by the present methods, there is a gap between the key values R_0^e and R_0^s, which means they are only sufficient conditions. We hope to establish an accurate sufficient and necessary condition for disease extinction and persistence which is our working towards.

Declarations

Ethical Approval: Not applicable.

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