GENERAL DECAY RESULT FOR A VON KARMAN EQUATIONS WITH MEMORY

Min Yoon[1], Jum-Ran Kang[2,*]

Department of Applied Mathematics, Pukyong National University, Busan 48513, South Korea

Abstract. This paper is concerned with a von Karman plate model with memory. Using some properties of the convex function and the multiplier method, we show the general decay rate result for a von Karman equations with minimal condition on the relaxation function. This result extends and improves on some earlier results-exponential or polynomial decay rates for a von Karman equations with memory.

Keywords: von Karman equations, viscoelastic, general decay rate, convexity, relaxation function

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1. Introduction

This paper is concerned with the general decay of the solutions to a von Karman equations with memory:

$$w_{tt} - \alpha \Delta w_{tt} + \Delta^2 w - \int_0^t h(t-s)\Delta^2 w(s)ds = [w, v] \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$\Delta^2 v = -[w, w] \quad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0,\infty), \tag{1.3}$$

$$w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{1.4}$$

$$\mathcal{B}_1 w - \mathcal{B}_1\Big\{\int_0^t h(t-s)w(s)ds\Big\} = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.5}$$

$$\mathcal{B}_2 w - \alpha\frac{\partial w_{tt}}{\partial \nu} - \mathcal{B}_2\Big\{\int_0^t h(t-s)w(s)ds\Big\} = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.6}$$

$$w(x,y,0) = w_0(x,y), \ \ w_t(x,y,0) = w_1(x,y) \quad \text{in } \Omega. \tag{1.7}$$

where w is the vertical displacement and v is the Airy-stress function. The von Karman equations describe small vibrations of a thin isotropic plate of uniform thickness α. Here Ω is an open bounded set of $\mathbb{R}^2$ with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, Γ_0 and Γ_1 are closed and disjoint. The relaxation function h is a positive decreasing function and we will give later the minimal conditions on h in order to obtain the general decay results. The differential operators

$$\mathcal{B}_1 w = \Delta w + (1-\mu)B_1 w, \quad \text{and} \quad \mathcal{B}_2 w = \frac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w$$

Tel.: +82 51 629 5523.
E-mail address: myoon@pknu.ac.kr[1], jrkang@pknu.ac.kr[2].
[*] corresponding author.

where constant $\mu(0 < \mu < \frac{1}{2})$ is Poisson's ratio and

$$B_1 w = 2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx}, \qquad B_2 w = \frac{\partial}{\partial\eta}[(\nu_1^2 - \nu_2^2)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})].$$

Here $\nu = (\nu_1, \nu_2)$ is the outward unit normal vector to Γ, $\eta = (-\nu_2, \nu_1)$ is the corresponding unit tangent vector. The von Kármán bracket is given by

$$[w, u] = w_{xx}u_{yy} - 2w_{xy}u_{xy} + w_{yy}u_{xx}.$$

The energy decay of the solutions to a von Karman system has been studied by several authors. In [1-3] the authors considered the von Karman system with frictional dissipations effective in the boundary. Rivera and Menzala [4] proved the stability of the solutions to a von Karman system for viscoelastic plates with boundary memory conditions. They obtained that the energy decays uniformly exponentially or algebraically with the same rate of decay as the relaxation function. The function h satisfies

$$-C_0 h(t) \leq h'(t) \leq -C_1 h(t) \text{ and } 0 \leq h''(t) \leq C_2 h(t)$$

for some $C_i, i = 0, 1, 2$. Raposo and Santos [5] improved the decay result of [4]. They proved the general decay of the solutions to a von Karman plate model under the condition on h such as

$$h'(t) \leq -\xi(t)h(t), \quad \xi(t) > 0, \quad \xi'(t) < 0, \ \forall t \geq 0 \tag{1.8}$$

where ξ is a nonincreasing and positive function. Kang [6] showed the general decay of the solutions to a von Karman plate model with memory and boundary damping. Kang [6] generalized the results of [5] without imposing any restrictive growth assumption on the damping term. Kang [7] studied the general stability for a von Karman system with memory using some properties of the convex functions. The relaxation function h satisfies

$$h'(t) \leq -H(h(t)), \tag{1.9}$$

where H is a non-negative function, with $H(0) = 0$, and H is a linear or strictly increasing and strictly convex on $(0, r]$, for some $r > 0$. The above conditions are weaker conditions on H than those introduced in [8]. When $\alpha = 0$ in (1.1) and the memory kernel h satisfies (1.9), Cavalcanti et al. [9] studied the existence and uniform decay rates of the energy for solutions.

On the other hand, many authors([10-20]) investigated the energy decay rates of the solutions to a viscoelastic wave equation. Recently, Mustafa [21] considered the general decay rate for a viscoelastic wave equations under a more general condition than the ones in (1.8) and (1.9) such as

$$h'(t) \leq -\xi(t)H(h(t)), \tag{1.10}$$

where ξ is a positive nonincreasing differentiable function and H is a non-negative function, with $H(0) = 0$, and H is a linear or strictly increasing and strictly convex on $(0, r]$, for some $0 < r \leq h(0)$. When h satisfies the condition (1.10), the stability of the solutions to a viscoelastic system was studied in [22-24] and the references therein.

Motivated by the work in [21], we prove the general decay of the solutions to a von Karman plate model (1.1)-(1.7) for relaxation function h satisfying (1.10). Using the multiplier method and some properties of convex functions, we establish the general decay rate of the solutions to a von Karman plate model (1.1)-(1.7).

The paper is organized as follows. In Section 2, we present some notations and material needed for our work and state the main result. In Section 3, we prove the general decay of the solutions to the von Karman system with general type of relaxation functions.

2. PRELIMINARIES

In this section, we present some material needed in the proof of our result and state the main result. Throughout this paper we define

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0\},\ W = \Big\{w \in H^2(\Omega) \mid w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0\Big\}.$$

For simplicity, we denote $||\cdot||_{L^2(\Omega)}$ by $||\cdot||$. From the Green's formula, we have

$$(\Delta^2 w, u) = a(w,u) + (\mathcal{B}_2 w, u)_\Gamma - \Big(\mathcal{B}_1 w, \frac{\partial u}{\partial \nu}\Big)_\Gamma \tag{2.1}$$

where the bilinear symmetric form $a(w,u)$ is given by

$$a(w,u) = \int_\Omega \Big\{w_{xx}u_{xx} + w_{yy}u_{yy} + \mu(w_{xx}u_{yy} + w_{yy}u_{xx}) + 2(1-\mu)w_{xy}u_{xy}\Big\}d\Omega,$$

where $d\Omega = dxdy$. Since $\Gamma_0 \neq \emptyset$, we find that $\sqrt{a(w,w)}$ is equivalent to the $H^2(\Omega)$ norm on W, that is,

$$c_0||w||^2_{H^2(\Omega)} \le a(w,w) \le c_1||w||^2_{H^2(\Omega)}, \tag{2.2}$$

where c_0 and c_1 are positive constants. The Sobolev imbedding theorem and (2.2) imply that

$$||w||^2 \le c_p a(w,w),\ \ ||\nabla w||^2 \le c_s a(w,w),\ \forall w \in W \tag{2.3}$$

where c_p and c_s are positive constants. Using the symmetry of $a(\cdot,\cdot)$, we obtain that for any $w \in C^1(0,T;H^2(\Omega))$,

$$a(h*w, w_t) = -\frac{1}{2}\frac{d}{dt}\Big\{h\Box\partial^2 w - \Big(\int_0^t h(s)ds\Big)a(w,w)\Big\} - \frac{1}{2}h(t)a(w,w) + \frac{1}{2}h'\Box\partial^2 w, \tag{2.4}$$

where

$$(h*w)(t) := \int_0^t h(t-s)w(s)ds,\ \ (h\Box\partial^2 w)(t) := \int_0^t h(t-s)a(w(\cdot,t)-w(\cdot,s), w(\cdot,t)-w(\cdot,s))ds.$$

We consider the following hypotheses:

(H1) $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$1 - \int_0^\infty h(s)ds = l > 0. \tag{2.5}$$

(H2) There exists a positive function $G \in C^1(\mathbb{R}^+)$, with $G(0) = G'(0) = 0$, and G is a linear or it is strictly increasing and strictly convex C^2 function on $(0,k],\ k \le h(0)$, such that

$$h'(t) \le -\zeta(t)G(h(t)),\ \ \forall t \ge 0, \tag{2.6}$$

where ζ is a positive nonincreasing differentiable function.

Specific examples of relaxation functions are introduced in reference [21].

Remark 2.1. ([21])

1. By using (H1) and (H2), we conclude that $\lim_{t\to+\infty} h(t) = 0$. Therefore, there is $t_0 > 0$ large enough such that

$$h(t_0) = k \implies h(t) \le k, \quad \forall t \ge t_0.$$

Because h and ζ are positive nonincreasing continuous functions and G is a positive continuous function then, for all $t \in [0, t_0]$,

$$h'(t) \le -\zeta(t)G(h(t)) \le -\frac{c_2}{h(0)}h(0) \le -\frac{c_2}{h(0)}h(t),$$

which gives

$$h'(t) \le -c_3 h(t), \quad \forall t \in [0, t_0], \tag{2.7}$$

where c_2 and c_3 are positive constants.

2. If G is a strictly convex on $(0, k]$ and $G(0) = 0$, then

$$G(\theta x) \le \theta G(x), \quad x \in (0, k] \text{ and } 0 \le \theta \le 1. \tag{2.8}$$

3. Let G^* be the convex conjugate of G in the sense of Young (see [25]); then

$$G^*(s) = s(G')^{-1}(s) - G[(G')^{-1}(s)] \le s(G')^{-1}(s), \quad \text{if } s \in (0, G'(k)] \tag{2.9}$$

and G^* satisfies the following Young's inequality

$$AB \le G^*(A) + G(B), \quad \text{if } A \in (0, G'(k)],\ B \in (0, k]. \tag{2.10}$$

The well-known Jensen's inequality and lemma for the bracket's binary will be of essential use in proving our result.

Remark 2.2. (Jensen's inequality) If P is a convex function on $[a, b]$, $\xi : \Omega \to [a, b]$ and q are integrable functions on Ω, $q(x) \ge 0$, and $\int_\Omega q(x)dx = q_0 > 0$, then Jensen's inequality states that

$$P\Big(\frac{1}{q_0}\int_\Omega \xi(x)q(x)dx\Big) \le \frac{1}{q_0}\int_\Omega P\big(\xi(x)\big)q(x)dx. \tag{2.11}$$

Lemma 2.1. ([9]) The bilinear form

$$[\cdot,\cdot] : H_0^2(\Omega) \times H_0^2(\Omega) \to H^{-1-\epsilon}(\Omega)$$
$$(w, v) \longmapsto [w, v]$$

is continuous for every $\epsilon > 0$. In addition, the following estimate holds

$$||[w, v]||_{H^{-1-\epsilon}(\Omega)} \le c_4 ||w||_{H_0^2(\Omega)} ||v||_{H_0^2(\Omega)}, \tag{2.12}$$

where c_4 is a positive constant.

Using the Galerkin's approximation, we can prove the following result of the solution for a von Karman plate model with memory (1.1)-(1.7).

Theorem 2.1. ([4, 5]) For the initial data $(w_0, w_1) \in W \times V$, $T > 0$ and $\alpha > 0$, the system (1.1)-(1.7) has a unique weak solution. For $(w_0, w_1) \in (W \cap H^4(\Omega)) \times (V \cap H^3(\Omega))$, the weak solution satisfies

$$w \in C^0([0,T]; W \cap H^4(\Omega)) \cap C^1([0,T]; V \cap H^3(\Omega)).$$

Let us introduce the energy of problem (1.1)-(1.7)

$$E(t) = \frac{1}{2}||w_t||^2 + \frac{\alpha}{2}||\nabla w_t||^2 + \frac{1}{2}\Big(1 - \int_0^t h(s)ds\Big)a(w,w) + \frac{1}{2}h\Box\partial^2 w + \frac{1}{4}||\Delta v||^2. \tag{2.13}$$

Now, we state the main result.

Theorem 2.2. Assume that (H1) and (H2) hold. Then there exist positive constants k_0 and k_1 such that the energy functional satisfies

$$E(t) \leq k_1 G_1^{-1}\Big(k_0 \int_{h^{-1}(k)}^t \zeta(s)ds\Big) \tag{2.14}$$

where

$$G_1(t) = \int_t^k \frac{1}{sG'(s)}ds$$

and G_1 is strictly decreasing and convex on $(0, k]$, with $\lim_{t\to 0} G_1(t) = +\infty$.

3. General decay of the energy

In this section, we prove the general decay rates in Theorem 2.2. First, Multiplying (1.1) by $w_t(t)$ and using (2.1), (2.4) and (2.13), we get

$$E'(t) = -\frac{h(t)}{2}a(w,w) + \frac{1}{2}h'\Box\partial^2 w \leq 0. \tag{3.1}$$

This implies that $E(t)$ is nonincreasing.

For suitable choice of $N, N_0, N_1 > 0$, let us define the perturbed energy by

$$L(t) = NE(t) + N_0\Phi(t) + N_1\Psi(t), \tag{3.2}$$

where

$$\Phi(t) = \int_\Omega w_t w d\Omega + \alpha \int_\Omega \nabla w_t \nabla w d\Omega$$

and

$$\Psi(t) = \int_\Omega (\alpha\Delta w_t - w_t)\int_0^t h(t-s)(w(t) - w(s))dsd\Omega.$$

By the ideas presented in [6, 7], we easily have the following lemma.

Lemma 3.1. For $N > 0$ large enough, there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t), \quad \forall t \geq 0. \tag{3.3}$$

Lemma 3.2. Under the assumption (H1), the functional $\Phi(t)$ satisfies, along the solution of (1.1)-(1.7), the estimate

$$\Phi'(t) \leq -\frac{l}{2}a(w,w) + ||w_t||^2 + \alpha||\nabla w_t||^2 + \frac{C_\gamma}{2l}f\Box\partial^2 w - ||\Delta v||^2 \tag{3.4}$$

for any $0 < \gamma < 1$, where

$$C_\gamma = \int_0^\infty \frac{h^2(s)}{f(s)}ds \quad \text{and} \quad f(t) = \gamma h(t) - h'(t) > 0. \tag{3.5}$$

Proof. From (1.1), (2.5) and Young's inequality, we obtain

$$\begin{aligned}
\Phi'(t) &= -a(w,w) + a(h*w,w) + ([w,v],w) + ||w_t||^2 + \alpha||\nabla w_t||^2 \\
&\leq ||w_t||^2 + \alpha||\nabla w_t||^2 - ||\Delta v||^2 - \Big(1 - \int_0^\infty h(s)ds\Big)a(w,w) + \int_0^t h(t-s)a(w(s)-w(t),w(t))ds \\
&\leq ||w_t||^2 + \alpha||\nabla w_t||^2 - ||\Delta v||^2 - \frac{l}{2}a(w,w) \\
&+\frac{1}{2l}\int_0^t h(t-s)\int_0^t h(t-s)a(w(t)-w(s),w(t)-w(s))dsds.
\end{aligned} \tag{3.6}$$

By using Cauchy-Schwarz inequality and (3.5), we have

$$\begin{aligned}
&\int_0^t h(t-s)\int_0^t h(t-s)a(w(t)-w(s),w(t)-w(s))dsds \\
&\leq \Big(\int_0^t \frac{h^2(s)}{f(s)}ds\Big)\int_0^t f(t-s)a(w(t)-w(s),w(t)-w(s))ds \leq C_\gamma f\Box\partial^2 w.
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we get the estimate (3.4). $\Box$

Lemma 3.3. Under the assumption (H1), the functional $\Psi(t)$ satisfies, along the solution of (1.1)-(1.7), the estimate

$$\Psi'(t) \leq -\Big(\int_0^t h(s)ds - \delta\Big)\big(||w_t||^2 + \alpha||\nabla w_t||^2\big) + \delta a(w,w) + c_5\delta||\Delta v||^2 + \Big(C_\gamma + \frac{c_6 + c_7C_\gamma}{2\delta}\Big)f\Box\partial^2 w, \tag{3.8}$$

where $0 < \delta < 1$, $c_5 = \frac{4E(0)^2}{lc_0^2}$, $c_6 = (c_p + \alpha c_s)\big(h(0) + \gamma(1-l)\big)$ and $c_7 = 1 + \frac{c_4^2}{2c_0} + \gamma^2(c_p + \alpha c_s)$.

Proof. Similarly we see that

$$\begin{aligned}
\Psi'(t) &= \Big(1 - \int_0^t h(s)ds\Big)\int_0^t h(t-s)a(w(t)-w(s),w(t))ds - \int_0^t h(t-s)(w(t)-w(s),[w,v])ds \\
&+ \int_0^t h(t-s)a(w(t)-w(s),\int_0^t h(t-s)(w(t)-w(s))ds)ds \\
&- \int_0^t h'(t-s)(w(t)-w(s),w_t(t))ds - \Big(\int_0^t h(s)ds\Big)||w_t||^2 \\
&-\alpha\int_0^t h'(t-s)(\nabla w(t)-\nabla w(s),\nabla w_t(t))ds - \alpha\Big(\int_0^t h(s)ds\Big)||\nabla w_t||^2 \\
&:= I_1 + I_2 + \cdots + I_5 - \Big(\int_0^t h(s)ds\Big)||w_t||^2 - \alpha\Big(\int_0^t h(s)ds\Big)||\nabla w_t||^2.
\end{aligned} \tag{3.9}$$

Now, we estimate the terms in the right hand side of (3.9). Using Young and Hölder inequalities, (2.2), (2.12), (2.13), (3.1)and (3.7), we have

$$\begin{aligned}
|I_1| &\le \frac{\delta}{2}a(w,w) + \frac{1}{2\delta}\int_0^t h(t-s)\int_0^t h(t-s)a(w(t)-w(s),w(t)-w(s))dsds\\
&\le \frac{\delta}{2}a(w,w) + \frac{C_\gamma}{2\delta}f\Box\partial^2 w,\\
|I_2| &= \Big|\int_\Omega [w,v]\int_0^t h(t-s)(w(t)-w(s))dsd\Omega\Big| \le c_4||\Delta w||||\Delta v||\Big|\Big|\int_0^t h(t-s)(\Delta w(t)-\Delta w(s))ds\Big|\Big|\\
&\le \delta||\Delta w||^2||\Delta v||^2 + \frac{c_4^2}{4\delta}\Big|\Big|\int_0^t h(t-s)(\Delta w(t)-\Delta w(s))ds\Big|\Big|^2\\
&\le \frac{\delta l c_0^2}{4E(0)}||\Delta w||^4 + \frac{\delta E(0)}{lc_0^2}||\Delta v||^4 + \frac{c_4^2}{4\delta}\Big|\Big|\int_0^t h(t-s)(\Delta w(t)-\Delta w(s))ds\Big|\Big|^2\\
&\le \frac{\delta}{2}a(w,w) + \frac{4E(0)^2\delta}{lc_0^2}||\Delta v||^2 + \frac{c_4^2 C_\gamma}{4\delta c_0}f\Box\partial^2 w,\\
|I_3| &\le C_\gamma f\Box\partial^2 w.
\end{aligned}$$

By the Young's inequality, Cauchy-Schwarz inequality, (2.3) and (3.5), we obtain

$$\begin{aligned}
|I_4| &\le \Big|\int_0^t f(t-s)(w(t)-w(s),w_t(t))ds\Big| + \Big|\gamma\int_0^t h(t-s)(w(t)-w(s),w_t(t))ds\Big|\\
&\le \delta||w_t||^2 + \frac{1}{2\delta}\int_\Omega\Big(\int_0^t f(t-s)|w(t)-w(s)|ds\Big)^2dx + \frac{\gamma^2}{2\delta}\int_\Omega\Big(\int_0^t h(t-s)|w(t)-w(s)|ds\Big)^2dx\\
&\le \delta||w_t||^2 + \frac{c_p}{2\delta}\Big(\int_0^t f(s)ds\Big)f\Box\partial^2 w + \frac{\gamma^2}{2\delta}\Big(\int_0^t \frac{h^2(s)}{f(s)}ds\Big)\int_\Omega\int_0^t f(t-s)|w(t)-w(s)|^2dsdx\\
&\le \delta||w_t||^2 + \frac{c_p\big(h(0)+\gamma(1-l)+\gamma^2C_\gamma\big)}{2\delta}f\Box\partial^2 w,\\
|I_5| &\le \alpha\delta||\nabla w_t||^2 + \frac{\alpha c_s\big(h(0)+\gamma(1-l)+\gamma^2C_\gamma\big)}{2\delta}f\Box\partial^2 w.
\end{aligned}$$

From all above estimates and (3.9), we arrive at

$$\begin{aligned}
\Psi'(t) \le& -\Big(\int_0^t h(s)ds-\delta\Big)||w_t||^2 - \Big(\int_0^t h(s)ds-\delta\Big)\alpha||\nabla w_t||^2 + \delta a(w,w) + \frac{4E(0)^2\delta}{lc_0^2}||\Delta v||^2\\
&+\Big(C_\gamma + \frac{C_\gamma}{2\delta} + \frac{c_4^2C_\gamma}{4\delta c_0} + \frac{(c_p+\alpha c_s)\big(h(0)+\gamma(1-l)+\gamma^2C_\gamma\big)}{2\delta}\Big)f\Box\partial^2 w.
\end{aligned}$$

□

Now, we establish the estimate of the Lyapunov functional L.

Lemma 3.4. Under the assumptions (H1) and (H2). Then for suitable choice of $N, N_0, N_1 > 0$, the functional L satisfies that

$$L'(t) \le -||w_t||^2 - \alpha||\nabla w_t||^2 - 6(1-l)a(w,w) - ||\Delta v||^2 + \frac{1}{2}h\Box\partial^2 w, \quad \forall t \ge t_0. \tag{3.10}$$

Proof. Since h is positive, we get $\int_0^t h(s)ds \geq h_0$, for all $t \geq t_0$. Thus, making use of this and combining (3.1), (3.2), (3.4), (3.8), recalling that $h'(t) = \gamma h(t) - f(t)$, and choosing $\delta = \frac{l}{2N_1}$, we have

$$L'(t) \leq -\Big(N_1 h_0 - N_0 - \frac{l}{2}\Big)||w_t||^2 - \alpha\Big(N_1 h_0 - N_0 - \frac{l}{2}\Big)||\nabla w_t||^2 - \Big(\frac{lN_0}{2} - \frac{l}{2}\Big)a(w,w)$$
$$-\big(N_0 - \frac{lc_5}{2}\big)||\Delta v||^2 + \frac{\gamma N}{2} h\Box\partial^2 w - \Big(\frac{N}{2} - \frac{c_6 N_1^2}{l} - C_\gamma\big(\frac{N_0}{2l} + \frac{c_7 N_1^2}{l} + N_1\big)\Big) f\Box\partial^2 w.$$

We first take N_0 large enough so that

$$\frac{lN_0}{2} - \frac{l}{2} > 6(1-l), \quad N_0 - \frac{lc_5}{2} > 1 \tag{3.11}$$

then N_1 large enough so that

$$N_1 h_0 - N_0 - \frac{l}{2} > 1. \tag{3.12}$$

From (2.6) and (3.5), we get

$$0 \leq -h'(t) \quad \Rightarrow \quad \gamma h(t) \leq \gamma h(t) - h'(t) \quad \Rightarrow \quad \frac{\gamma h(t)}{f(t)} \leq 1 \quad \Rightarrow \quad \frac{\gamma h^2(t)}{f(t)} \leq h(t). \tag{3.13}$$

Therefore, by (2.5) and (3.13), we obtain

$$\gamma C_\gamma = \gamma \int_0^\infty \frac{h^2(s)}{f(s)} ds \leq \int_0^\infty h(s)ds = 1 - l. \tag{3.14}$$

Applying (3.14) and the Lebesgue dominated convergence theorem, we have

$$\gamma C_\gamma \to 0 \ \text{ as } \ \gamma \to 0.$$

Hence, there is $0 < \gamma_0 < 1$ such that if $\gamma < \gamma_0$, then

$$\gamma C_\gamma\big(\frac{N_0}{2l} + \frac{c_7 N_1^2}{l} + N_1\big)\Big) < \frac{1}{4}.$$

We choose $\gamma = \frac{1}{N} < \gamma_0$ and take N large enough so that

$$\frac{N}{4} - \frac{c_6 N_1^2}{l} > 0,$$

which means

$$\frac{N}{2} - \frac{c_6 N_1^2}{l} - C_\gamma\big(\frac{N_0}{2l} + \frac{c_7 N_1^2}{l} + N_1\big) > 0. \tag{3.15}$$

Combining (3.11), (3.12) and (3.15) gives (3.10). □

Lemma 3.5. Under the assumption (H1), the functional K defined by

$$K(t) = \int_0^t g(t-s)a(w(s), w(s))ds$$

satisfies the estimate

$$K'(t) \leq 5(1-l)a(w,w) - \frac{3}{4} h\Box\partial^2 w, \tag{3.16}$$

where $g(t) = \int_t^\infty h(s)ds$.

Proof. From $g'(t) = -h(t)$, we have $g(t) - g(0) = -\int_0^t h(s)ds \le 0$, which means

$$g(t) \le g(0) = \int_0^\infty h(s)ds = 1 - l. \tag{3.17}$$

Using the Young's inequality, (2.5) and (3.17), we see that

$$\begin{aligned} K'(t) &= g(0)a(w,w) - \int_0^t h(t-s)a(w(s),w(s))ds \\ &\le \Big(g(0) - \int_0^t h(s)ds\Big)a(w,w) - h\Box\partial^2 w - 2\int_0^t h(t-s)a(w(s)-w(t),w(t))ds \\ &\le 5(1-l)a(w,w) - \frac{3}{4}h\Box\partial^2 w. \end{aligned}$$

□

Proof of Theorem 2.2. From (2.13) and (3.10), there exist positive constants β_1 and β_2 such that

$$L'(t) \le -\beta_1 E(t) + \beta_2 h\Box\partial^2 w. \tag{3.18}$$

Using (2.7), (3.1) and (3.18), we get, for all $t \ge t_0$,

$$\mathcal{L}'(t) \le -\beta_1 E(t) + \beta_2 \int_{t_0}^t h(s)a(w(t)-w(t-s), w(t)-w(t-s))ds \tag{3.19}$$

where $\mathcal{L}(t) = L(t) + \frac{2\beta_2}{c_3}E(t)$, which is clearly equivalent to $E(t)$.

We consider the following two cases.

Case 1. $G(t)$ is linear: Multiplying (3.19) by $\zeta(t)$ and utilizing (2.6) and (3.1), we have

$$\begin{aligned} \zeta(t)\mathcal{L}'(t) &\le -\beta_1\zeta(t)E(t) - \beta_2\int_{t_0}^t h'(s)a(w(t)-w(t-s), w(t)-w(t-s))ds \\ &\le -\beta_1\zeta(t)E(t) - 2\beta_2 E'(t), \end{aligned} \tag{3.20}$$

which gives

$$\big(\zeta\mathcal{L} + 2\beta_2 E\big)'(t) \le -\beta_1\zeta(t)E(t), \quad \forall t \ge t_0.$$

From (3.3), we find that $\zeta\mathcal{L} + 2\beta_2 E \sim E$. Then, we obtain

$$E(t) \le c' e^{-c\int_{t_0}^t \zeta(s)ds}.$$

Case 2. $G(t)$ is nonlinear: This case is obtained on account of the ideas presented in [21, 23, 24] as follows. We take

$$F(t) = L(t) + K(t),$$

which is nonnegative. From (2.13), (3.10) and (3.16), we get

$$F'(t) \le -||w_t||^2 - \alpha||\nabla w_t||^2 - (1-l)a(w,w) - \frac{1}{4}h\Box\partial^2 w - ||\Delta v||^2 \le -\beta_3 E(t) \tag{3.21}$$

where β_3 is some positive constant. Integrating (3.21) from t_0 to t, we obtain

$$\beta_3\int_{t_0}^t E(s)ds \le F(t_0) - F(t) \le F(t_0).$$

Therefore, we deduce that

$$\int_0^\infty E(s)ds < \infty. \tag{3.22}$$

We define $\eta(t)$ by, for a constant $0 < p < 1$,

$$\eta(t) := p\int_{t_0}^t a(w(t) - w(t-s), w(t) - w(t-s))ds.$$

From (3.22), we find that $\eta(t)$ satisfies

$$\eta(t) < 1, \ \forall t \geq t_0. \tag{3.23}$$

Using (2.6), (2.8), (2.11), (3.1), (3.23) and the fact that $\zeta(t)$ is a positive nonincreasing function, we find that

$$\begin{aligned}
&\int_{t_0}^t h(s)a(w(t) - w(t-s), w(t) - w(t-s))ds \\
&\leq \frac{\eta(t)}{p}\int_{t_0}^t G^{-1}\Big(-\frac{h'(s)}{\zeta(s)}\Big)\frac{pa(w(t) - w(t-s), w(t) - w(t-s))}{\eta(t)}ds \\
&\leq \frac{\eta(t)}{p}G^{-1}\left(p\int_{t_0}^t \frac{-h'(s)a(w(t) - w(t-s), w(t) - w(t-s))}{\zeta(s)\eta(t)}ds\right) \\
&\leq \frac{1}{p}G^{-1}\left(p\int_{t_0}^t \frac{-h'(s)a(w(t) - w(t-s), w(t) - w(t-s))}{\zeta(s)}ds\right) \\
&\leq \frac{1}{p}G^{-1}\Big(\frac{1}{\zeta(t)}\int_{t_0}^t -h'(s)a(w(t) - w(t-s), w(t) - w(t-s))ds\Big) \\
&\leq \frac{1}{p}G^{-1}\Big(\frac{-2E(t)}{\zeta(t)}\Big).
\end{aligned} \tag{3.24}$$

Combining (3.19) and (3.24), we have

$$\mathcal{L}'(t) \leq -\beta_1 E(t) + \frac{\beta_2}{p}G^{-1}\Big(\frac{-2E(t)}{\zeta(t)}\Big), \quad \forall t \geq t_0. \tag{3.25}$$

Now, for $\epsilon_0 < \frac{k}{E(0)}$, we define the functional

$$I(t) := \mathcal{L}(t)G'\big(\epsilon_0 E(t)\big) + E(t),$$

which is equivalent to E. With $A = G'\big(\epsilon_0 E(t)\big)$ and $B = G^{-1}\Big(\frac{-2E(t)}{\zeta(t)}\Big)$, using (2.9), (2.10), (3.25) and the fact that $E' \leq 0, G > 0, G' > 0$ and $G'' > 0$, we see that

$$\begin{aligned}
I'(t) &\leq -\beta_1 E(t)G'\big(\epsilon_0 E(t)\big) + \frac{\beta_2}{p}G'\big(\epsilon_0 E(t)\big)G^{-1}\Big(\frac{-2E(t)}{\zeta(t)}\Big) \\
&\leq -\beta_1 E(t)G'\big(\epsilon_0 E(t)\big) + \frac{\beta_2}{p}G^*\Big(G'\big(\epsilon_0 E(t)\big)\Big) - \frac{2\beta_2 E'(t)}{p\zeta(t)} \\
&\leq -\Big(\beta_1 - \frac{\epsilon_0\beta_2}{p}\Big)E(t)G'\big(\epsilon_0 E(t)\big) - \frac{2\beta_2 E'(t)}{p\zeta(t)}.
\end{aligned} \tag{3.26}$$

We take $R(t) = \zeta(t)I(t) + \frac{2\beta_2 E(t)}{p}$, which satisfies

$$d_1 E(t) \leq R(t) \leq d_2 E(t), \tag{3.27}$$

for some $d_1, d_2 > 0$. Consequently, with a suitable choice of ϵ_0, and using (3.26), (3.27) and the fact that $\zeta(t)$ is a positive nonincreasing function and $G'(t)$ is strictly increasing function, we arrive at

$$R'(t) \le -\Big(\beta_1 - \frac{\epsilon_0\beta_2}{p}\Big)\zeta(t)E(t)G'\big(\epsilon_0 E(t)\big) \le -k_0\zeta(t)R(t)G'(d_3R(t)), \ \forall t \ge t_0,$$

where $k_0 = \frac{1}{d_2}\Big(\beta_1 - \frac{\epsilon_0\beta_2}{p}\Big) > 0$ and $d_3 = \frac{\epsilon_0}{d_2}$. Hence, a simple integration and a variable transformation give

$$\int_{t_0}^{t} \frac{-R'(s)}{R(s)G'(d_3R(s))}ds \ge k_0\int_{t_0}^{t}\zeta(s)ds \implies \int_{d_3R(t)}^{d_3R(t_0)} \frac{1}{sG'(s)}ds \ge k_0\int_{t_0}^{t}\zeta(s)ds, \quad \forall t \ge t_0. \tag{3.28}$$

We take

$$G_1(t) = \int_t^k \frac{1}{sG'(s)}ds,$$

which is strictly decreasing function on $(0, k]$. From (3.27), (3.28) and $\epsilon_0 E(0) < k$, we have

$$G_1(d_3R(t)) = \int_{d_3R(t)}^{k} \frac{1}{sG'(s)}ds \ge \int_{d_3R(t)}^{d_3R(t_0)} \frac{1}{sG'(s)}ds \ge k_0\int_{t_0}^{t}\zeta(s)ds, \quad \forall t \ge t_0.$$

Applying (3.27) and the fact that G_1 is strictly decreasing function on $(0, k]$, we conclude that

$$E(t) \le k_1 G_1^{-1}\Big(k_0\int_{t_0}^{t}\zeta(s)ds\Big)$$

where $k_1 = \frac{1}{d_1d_3} > 0$. Therefore, estimate (2.14) is established.

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Conflicts of interest

This work does not have any conflicts of interest.

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