

RESEARCH ARTICLES

Disturbance observer-based matrix-weighted consensus

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Abstract

In this paper, we consider the matrix-weighted consensus problems with disturbances. To this end, we firstly propose a new disturbance observer design for systems with unknown matched or mismatched disturbances representable as a linear combination of time-varying basis functions. Under some assumptions on the boundedness and persistent excitation of the regression matrix, the disturbances can be precisely estimated at an exponential rate and thus, can be compensated by a suitable compensation input. Next, disturbance-observer based consensus algorithms are proposed for matrix-weighted networks of single- and double-integrators with matched or mismatched disturbances. We show that both matched and mismatched disturbances can be estimated and actively compensated, and the system globally asymptotically converges to a fixed point in the kernel of the matrix-weighted Laplacian. Depending on the network connectivity, the system can asymptotically achieve a consensus or a cluster configuration. The disturbance-observer based consensus design is further extended for higher-order integrators with disturbances. Finally, simulation results are provided to support the mathematical analysis.

KEYWORDS

disturbance observer, consensus algorithm, matrix-weighted graphs

1 | INTRODUCTION

In the last decade, multi-agent consensus and clustering algorithms have been extensively studied. Several consensus-based applications such as formation control, network localization, or distributed computation, have been proposed and successfully realized¹.

With the rapidly theoretical developments of networked multi-agent systems, dynamical multi-dimensional networks have recently attracted a considerable amount of research attention². A multi-dimensional network contains several layers, where each node (agent) may interact with others via one or more layers. If the interlayer connections appear only between different layers of the same node, the network is called a *multiplex*, and can be considered as a collection of independent single-layer networks. The network becomes more complicated with cross-layer interactions (links between different layers of different nodes). An example of such *multi-layer network* is the multi-dimensional Friedkin-Johnsen opinion dynamics, where the multiple logically interdependent topics discussed by the agents are captured by the multi-issue dependence structure matrix^{3,4}. An attempt to model diffusive multi-dimensional networks with cross layer interactions was proposed in⁵, where agent-to-agent interactions were captured by positive definite/semidefinite matrix weights. Based on the matrix-weighted frameworks^{6,5}, different consensus problems were studied^{7,8,9,10} and several applications have been found, for examples, multi-dimensional opinion dynamics^{11,12}, synchronization of electrical/mechanical networks¹³, formation control and network localization¹⁴. To realize these proposed applications, the behaviors of the system under uncertain factors such as disturbances or parameter offsets needs to be considered, and corresponding robustified inputs are needed. To this end, back-stepping-based adaptive matrix-weighted consensus algorithms were proposed in¹⁵ for networks of high-order integrator agents with matched disturbances. In¹⁶, matrix-weighted consensus algorithms for networks of single- and double-integrator agents with time-varying bounded disturbances were designed by a sliding-mode control approach.

Abbreviations: Multi-agent systems (MASs), matrix-weighted graphs (MWGs), disturbance observer (DO).

In this paper, we considered the matrix-weighted consensus problem with both matched and mismatched disturbances from a disturbance observer perspective. Instead of considering uncertainties as unknown parts of the system and adjusting the algorithm's parameters accordingly as in^{15,16}, the deviation of the perturbed system with a nominal system is taken for estimating the unknown disturbance. Once the disturbances have been completely observed, a corresponding disturbance compensation action can be included to actively compensate the disturbance and eventually recover the nominal performance of the system^{17,18,19}. Disturbance observers have been shown to provide good performance in motion control and robotic manipulators^{20,21}. In addition, the information obtained by a disturbance observer can be used to detect abnormal state of the network. When the magnitude of the disturbance reaches a certain level, each agent may determine whether it is under attack and include a corresponding disturbance compensation input, or detect and temporally limit communication with a faulty neighboring agent^{22,23,24,25}.

The contributions of this paper are summarized as follows.

- We extend the disturbance observers in^{17,26} to deal with both matched/mismatched disturbances modeled by a constant vector left multiplied by a known time-varying matrix. Exponential stability of the proposed disturbance observers is proved under the assumption that the regressor matrices are bounded and persistently exciting. For related works, the disturbance observers in²⁶ were only applicable for second-order system with constant mismatched disturbance. The studies of exogenous state observers (ESO) usually assume matched disturbances generated by some known dynamics²⁷. Also, sliding-mode based observers, which can estimate bounded time-varying inputs in finite time via discontinuous estimation laws, can be found in²⁸. The proposed disturbance observers in this paper are continuous and do not require choosing some appropriate estimator's parameters as in the high-order sliding mode observers.
- We design matrix-weighted consensus algorithms for networks perturbed by disturbances based on the proposed disturbance observers. Firstly, we consider a matrix-weighted network of single-integrators perturbed by disturbances. Decentralized disturbance-observers are proposed for the system, which estimate the disturbance at an exponential rate. A corresponding disturbance compensator is then included to the consensus system to eliminate the disturbance. Thus, the network eventually converges to a fixed point in the kernel of the matrix-weighted Laplacian under the proposed algorithm. The fixed point may be a consensus point when the matrix-weighted Laplacian is of maximal rank, or may correspond to a clustering formation^{5,29}. Second, matrix-weighted consensus networks of double integrators with matched and mismatched disturbances are considered. A disturbance observer and a disturbance compensation input are proposed and their asymptotic behaviors are analysed accordingly. Finally, an extension to matrix weighted consensus network with higher-order integrators subjected to disturbances is considered. It is worth noting that disturbance observer for linear consensus systems interacted via a scalar weighted graph has been studied in³⁰. The work³⁰ assumed that the disturbances are matched and generated by exogenous dynamics. The authors in³¹ proposed two disturbance-observer-based consensus laws for linear agents with mismatched disturbances generated by an exogenous dynamics. The matrix gains of the first disturbance observer in³¹ were obtained by solving a matrix equation. The other dynamic compensator in³¹ was designed based on the agent's internal model. Disturbance observer based sliding-mode consensus laws for leader-follower higher-order integrator agents were proposed in³², which guarantee that the disturbance can be estimated in a finite time given that some control parameters are properly chosen. Another sliding-mode based distributed anti-disturbance output consensus algorithms for higher-order multi-agent systems with mismatched disturbances was proposed in³³, where the mismatched disturbances are assumed to have vanishing derivatives.

The rest of this paper is organized as follows. Section 2 presents several useful lemmas and preliminaries on matrix-weighted graphs and matrix-weighted consensus. A new class of disturbance observer is introduced in Section 3. In Section 4, disturbance observer-based matrix-weighted consensus laws are designed for multi-agent networks of single-, double-, and higher-order integrators with matched and mismatched disturbances. Simulation results are provided in Section 5, and Section 6 concludes the paper.

Notations. In this paper, vectors and matrices are denoted by bold font normal and capital letters, respectively, while scalars are denoted by normal font letters. The set of natural and real numbers are respectively denoted by \mathbb{N} and \mathbb{R} . The transpose of a matrix \mathbf{A} is denoted by \mathbf{A}^\top . Given a set of vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$, the vectorization operator is defined by $\text{vec}(\mathbf{x}_1, \dots, \mathbf{x}_n) = [\mathbf{x}_1^\top, \dots, \mathbf{x}_n^\top]^\top$. Given square matrices $\mathbf{X}_1, \dots, \mathbf{X}_n$, the matrix $\text{blkdiag}(\mathbf{X}_k)$ has $\mathbf{X}_1, \dots, \mathbf{X}_n$ as diagonal block matrices. The 1-norm, 2-norm, and ∞ -norm of a vector $\mathbf{y} = [y_1, \dots, y_d]^\top$ are $\|\mathbf{y}\|_1 = \sum_{k=1}^d |y_k|$, $\|\mathbf{y}\| = \sqrt{\sum_{k=1}^d y_k^2}$, and $\|\mathbf{y}\|_\infty = \max_{k=1, \dots, d} |y_k|$, respectively.

2 | PRELIMINARIES

2.1 | Stability of linear time-varying systems

In this subsection, we introduce two lemmas which will be used in the analysis. The first lemma is an consequence of input-to-state stability theory, and the second lemma is taken from adaptive control.

Lemma 1.³⁴ Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \varphi(t), \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^d$ is the state vector, $\mathbf{A} \in \mathbb{R}^{d \times d}$ is a Hurwitz matrix, and $\varphi(t)$ is a bounded vanishing input, $\lim_{t \rightarrow +\infty} \varphi(t) = \mathbf{0}_d$, to the system, then $\lim_{t \rightarrow +\infty} \mathbf{x}(t) = \mathbf{0}_d$, $\forall \mathbf{x}(0) \in \mathbb{R}^d$. Moreover, if there exist constants $C, \alpha > 0$ such that $\|\varphi(t)\| \leq Ce^{-\alpha t}$, then $\mathbf{x}(t)$ converges to $\mathbf{0}_d$ exponentially fast.

Lemma 2.^{35, Thm. 1} Consider the linear time-variant system of the form

$$\dot{\mathbf{x}}(t) = -\mathbf{Q}^\top(t)\mathbf{Q}(t)\mathbf{x}(t), \quad (2)$$

where $\mathbf{x} \in \mathbb{R}^r$ is a state variable vector, and $\mathbf{Q}(t)$ is a time-varying matrix. Suppose that $\mathbf{Q}(t)$ is persistently exciting, i.e., for some positive δ, α_1 , and α_2 , and $\forall t \geq 0$, there holds

$$\alpha_1 \mathbf{I}_r \leq \int_t^{t+\delta} \mathbf{Q}(t)^\top \mathbf{Q}(t) dt \leq \alpha_2 \mathbf{I}_r, \quad (3)$$

then the equilibrium $\mathbf{x} = \mathbf{0}_r$ of the system (2) is globally exponentially stable.

2.2 | Matrix-weighted networks

A matrix weighted graph is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, where $\mathcal{V} = \{1, \dots, n\}$, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and $\mathcal{W} = \{\mathbf{W}_{ij}\}_{(i,j) \in \mathcal{E}}$ are respectively the vertex set, the edge set, and the set of symmetric positive definite or positive semidefinite matrix weights. We assume that the graph is undirected ($\mathbf{W}_{ij} = \mathbf{W}_{ji}$) and contains no self-loop.

For each vertex $i \in \mathcal{V}$, the neighbor set of i and its matrix-weighted degree are respectively $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$ and $\mathbf{D}_i = \sum_{j \in \mathcal{N}_i} \mathbf{W}_{ji}$. The matrix-weighted adjacency-, degree-, and Laplacian matrices are respectively defined as $\mathbf{A} = [\mathbf{A}_{ij}] \in \mathbb{R}^{dn \times dn}$, $\mathbf{D} = \text{blkdiag}(\mathbf{D}_i) \in \mathbb{R}^{dn \times dn}$, $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{A}_{ij} = \mathbf{W}_{ji}$ if $(i, j) \in \mathcal{E}$ and $\mathbf{A}_{ij} = \mathbf{0}_{d \times d}$, otherwise. The matrix-weighted Laplacian \mathbf{L} of a undirected graph is symmetric positive semidefinite⁵ with eigenvalues $0 = \lambda_1 = \dots = \lambda_d \leq \lambda_{d+1} \leq \dots \leq dn$ and the corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_{dn}$. Thus, we can write $\mathbf{L} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top$, where $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_{dn}]$. Suppose that \mathbf{L} has 0 as a semisimple eigenvalue of multiple $l \geq d$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_l\}$ and $\{\mathbf{v}_{l+1}, \dots, \mathbf{v}_{dn}\}$ are the sets of normalized eigenvectors corresponding to the zero and positive eigenvalues of \mathbf{L} , respectively. These eigenvectors satisfy $\mathbf{v}_i^\top \mathbf{v}_j = 1$ if $i = j$ and $\mathbf{v}_i^\top \mathbf{v}_j = 0$ if $i \neq j$. Denoting $\mathbf{V}_1 = [\mathbf{v}_1, \dots, \mathbf{v}_l]$ and $\mathbf{V}_2 = [\mathbf{v}_{l+1}, \dots, \mathbf{v}_{dn}]$, it follows that $\mathbf{V}_k^\top \mathbf{V}_k = \mathbf{I}$ and $\mathbf{V}_k^\top \mathbf{V}_l = \mathbf{0}$, for $k, l = 1, 2, k \neq l$.

In this paper, we consider matrix-weighted networks consisting of p -th order integrators with matched or unmatched disturbances. The disturbances is estimated with observers and being compensated by an additional input. For all cases considered in this paper, the following assumptions are adopted.

Assumption 1. The matrix-weighted Laplacian matrix satisfies $\text{rank}(\mathbf{L}) = dn - l$, where $d \leq l < dn$.

For examples, for $l = d$, the kernel of \mathbf{L} is the consensus space $\text{im}(\mathbf{1}_n \otimes \mathbf{I}_n)$. For $l > d$, we can write $\ker(\mathbf{L}) = \{\text{im}[\mathbf{1}_n \otimes \mathbf{I}_d, \mathbf{v}_{d+1}^*, \dots, \mathbf{v}_l^*] | \mathbf{v}_k^* = \text{vec}(\mathbf{v}_{k1}^*, \dots, \mathbf{v}_{kn}^*), \mathbf{A}_{ij}(\mathbf{v}_{ki}^* - \mathbf{v}_{kj}^*) = \mathbf{0}_d, \forall (i, j) \in \mathcal{E}, \forall k = d+1, \dots, l\}$. Specially, for $l = d+1$, $\ker(\mathbf{L})$ contains all configurations which are bearing congruent to \mathbf{v}_{d+1}^* .[‡] We have the following lemma about the asymptotic behavior of the matrix-weighted consensus network without disturbance.

[‡] See ¹⁴ for a detailed introduction on bearing rigidity theory.

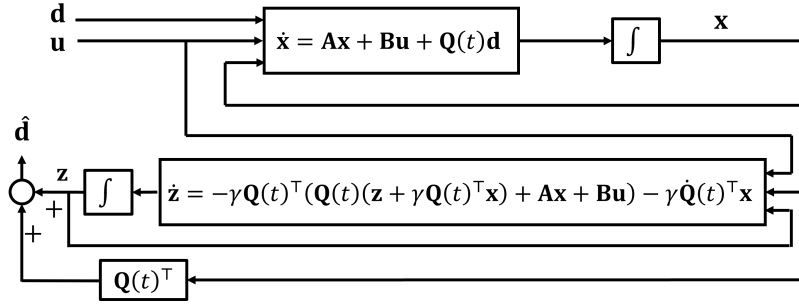


FIGURE 1 Block diagram describing the disturbance observer (7).

Lemma 3. Let each agent have a state vector $\mathbf{x}_i \in \mathbb{R}^d$ and update their states according to a single integrator dynamics. If $\text{rank}(\mathbf{L}) = dn - l$, where $d \leq l < dn$, then under the matrix-weighted consensus algorithm⁵

$$\dot{\mathbf{x}}_i(t) = - \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i(t) - \mathbf{x}_j(t)), \quad i = 1, \dots, n, \quad (4)$$

the state vector $\mathbf{x} = \text{vec}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ globally exponentially converges to a point in $\ker(\mathbf{L})$ determined by $\mathbf{x}^* = \mathbf{V}_1 \mathbf{V}_1^\top \mathbf{x}(0) = \sum_{i=1}^n \mathbf{v}_i \mathbf{v}_i^\top \mathbf{x}(0)$.

Proof. The system can be written in matrix form as

$$\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x} = -\mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top \mathbf{x}. \quad (5)$$

Since $\lambda_i = 0$, $i = 1, \dots, l$, $\mathbf{V}_1^\top \mathbf{x}(t) = \mathbf{V}_1^\top \mathbf{x}(0)$, $\forall t \geq 0$. Further, as $\lambda_i > 0$, $i = l + 1, \dots, dn$, we have $\mathbf{V}_2^\top \mathbf{x}(t)$ converges to $\mathbf{0}_{dn-l}$ exponentially fast. Thus, $\mathbf{x}(t)$ converges to $\mathbf{V}_1 \mathbf{V}_1^\top \mathbf{x}(0) = \sum_{i=1}^n \mathbf{v}_i \mathbf{v}_i^\top \mathbf{x}(0)$ at exponential rate. \square

3 | THE PROPOSED DISTURBANCE OBSERVERS BASED CONTROL LAW

3.1 | The disturbance observer

Consider a linear system with an additive disturbance in the following form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{Q}(t)\mathbf{d}, \quad (6)$$

where $\mathbf{x} \in \mathbb{R}^d$ is the state vector, $\mathbf{u} \in \mathbb{R}^h$ is the control input vector, $\mathbf{Q}(t) \in \mathbb{R}^{n \times r}$ is a known time-varying matrix, and $\mathbf{d} \in \mathbb{R}^r$ is a constant vector of unknown parameters, $\mathbf{A} \in \mathbb{R}^{d \times d}$ and $\mathbf{B} \in \mathbb{R}^{d \times h}$ are system matrices, and $d, r, h \in \mathbb{N}_+$. The disturbance term $\mathbf{Q}(t)\mathbf{d}$ is a matched disturbance if and only if the relation $\mathbf{Q}(t)\mathbf{d} \in \text{im}(\mathbf{B})$ for all $t \geq 0$, otherwise, it is a mismatched disturbance.

Remark 1. Let the matrix $\mathbf{Q}(t)$ be expressed as $\mathbf{Q}(t) = [\mathbf{q}_1(t), \dots, \mathbf{q}_r(t)]$. We can interpret each $\mathbf{q}_k(t)$, $k = 1, \dots, r$ as a possible disturbance function acting on the system. Then, $\mathbf{Q}(t)\mathbf{d} = \sum_{k=1}^r d_k \mathbf{q}_k(t)$ is the total disturbance. The unknown constants d_1, \dots, d_r are weights of the disturbance function. Thus, a disturbance observer is needed to estimate d_k based on observing the system's states, the control input, and the known vectors \mathbf{q}_k . The idea of having a regression matrix which is persistently exciting has a strong relation with classical adaptive control³⁶. The evolution of system's states generates useful information for determining unknown parameters. The process of estimating/learning a vector of unknown parameters corresponding to a set of possible candidate functions is also related to SINDy³⁷, a framework for system identification.

The disturbance observer is proposed as follows

$$\dot{\mathbf{z}} = -\gamma \mathbf{Q}^\top (\mathbf{Q}(\mathbf{z} + \gamma \mathbf{Q}^\top \mathbf{x}) + \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) - \gamma \dot{\mathbf{Q}}^\top \mathbf{x}, \quad (7a)$$

$$\hat{\mathbf{d}} = \mathbf{z} + \gamma \mathbf{Q}^\top \mathbf{x}, \quad (7b)$$

where $\gamma > 0$ is a constant. Figure 1 depicts the block diagram of the disturbance observer (7). We have the following theorem regarding the disturbance observer (7).

Theorem 1. Consider the system (6) with the disturbance observer (7) and let $\mathbf{e}_d = \hat{\mathbf{d}} - \mathbf{d}$. Then, $\mathbf{e}_d = \mathbf{0}_r$ is uniformly stable. Moreover, if $\mathbf{Q}(t)$ is persistently exciting, then $\mathbf{e}_d = \mathbf{0}_r$ is globally exponentially stable.

Proof. We have $\dot{\mathbf{e}}_d = \dot{\mathbf{z}} + \gamma \dot{\mathbf{Q}}^\top \mathbf{x} + \gamma \mathbf{Q}^\top (\mathbf{u} + \mathbf{Q}\mathbf{d}) = -\mathbf{Q}^\top \mathbf{Q}\mathbf{e}_d$. Consider the Lyapunov function $V = \frac{1}{2} \|\mathbf{e}_d\|^2$, which is continuously differentiable, positive definite, and radially unbounded. As $\dot{V} = -\mathbf{e}_d^\top \mathbf{Q}(t)^\top \mathbf{Q}(t) \mathbf{e}_d \leq 0$, it follows that \mathbf{e}_d is uniformly stable³⁴. If $\mathbf{Q}(t)$ satisfies the persistently exciting condition, the exponential stability of $\mathbf{e}_d = \mathbf{0}_r$ follows from Lemma 2. \square

3.2 | Disturbance observer-based control for high-order integrator system

In this subsection, we focus on designing disturbance observer-based control law for a high-order integrator system. This class of system arises in many applications of network systems such as consensus, formation control, or network localization.

3.2.1 | Matched disturbance

Consider a p -th order integrator system with matched disturbance

$$\dot{\mathbf{x}}^k = \mathbf{x}^{k+1}, \quad k = 1, \dots, p-1, \quad (8a)$$

$$\dot{\mathbf{x}}^p = \mathbf{u}(t) + \mathbf{Q}(t)\mathbf{d}, \quad (8b)$$

where $\mathbf{x}^k \in \mathbb{R}^d$, $k = 1, \dots, p$, are state vectors, the disturbance observer is designed as

$$\dot{\mathbf{z}} = -\gamma \mathbf{Q}^\top (\mathbf{Q}(\mathbf{z} + k \mathbf{Q}^\top \mathbf{x}^p) + \mathbf{u}) - \gamma \dot{\mathbf{Q}}^\top \mathbf{x}^p, \quad (9a)$$

$$\hat{\mathbf{d}} = \mathbf{z} + \gamma \mathbf{Q}^\top \mathbf{x}^p. \quad (9b)$$

We write $\mathbf{u}(t) = \mathbf{u}_c + \mathbf{w}$, where \mathbf{u}_c is designed for the control objective and \mathbf{w} is specified for rejecting the unknown disturbance $\mathbf{Q}(t)\mathbf{d}$. We have the following result.

Theorem 2. Consider the system (8) with the disturbance observer (9). Let $\mathbf{e}_d = \hat{\mathbf{d}} - \mathbf{d}$, and suppose that $\mathbf{Q}(t)$ is persistently exciting, then (i) $\mathbf{e}_d = \mathbf{0}_r$ is globally exponentially stable, (ii) let $\mathbf{u}_c = -\sum_{k=1}^p \gamma_k \mathbf{x}^k$, where $\gamma_k > 0$, $k = 1, \dots, p$ are chosen so that the polynomials $P(s) = s^p + \sum_{k=1}^p \gamma_k s^{k-1}$ is Hurwitz, and $\mathbf{w} = -\mathbf{Q}\hat{\mathbf{d}}$. Then, $\mathbf{x} = \text{vec}(\mathbf{x}^1, \dots, \mathbf{x}^p)$ converges to $\mathbf{0}_{dn}$ exponentially fast.

Proof. (i) The exponential convergence of $\mathbf{e}_d = \mathbf{0}_r$ follows directly from Thm. 1.

(ii) Next, the assumption that the polynomial $P(s)$ is Hurwitz implies that the nominal system of (9) (obtained by setting $\mathbf{w} = \mathbf{0}$ and $\mathbf{Q}(t)\mathbf{d} = \mathbf{0}$) is exponentially stable. Now, by inserting the compensation control input $\mathbf{w} = -\mathbf{Q}\hat{\mathbf{d}}$, the disturbance part acting on the system becomes $-\mathbf{Q}\mathbf{e}_d$, which converges to $\mathbf{0}$ exponentially fast due to (i). Based on Lemma 1, we conclude that $\mathbf{x} \rightarrow \mathbf{0}_{dn}$ exponentially fast. \square

3.3 | Mismatched disturbance observer based sliding-mode control for higher-order integrator

In this subsection, we consider the problem of stabilizing a p -th order integrator with mismatched disturbances

$$\dot{\mathbf{x}}^k = \mathbf{x}^{k+1} + \mathbf{Q}^k(t)\mathbf{d}^k, \quad k = 1, \dots, p-1, \quad (10a)$$

$$\dot{\mathbf{x}}^p = \mathbf{u}(t), \quad (10b)$$

where $\mathbf{x}^k \in \mathbb{R}^d$, $k = 1, \dots, p$, are state vectors, $\mathbf{d}^k \in \mathbb{R}^{r_k}$, $r_k \in \mathbb{N}$, $r_k > 0$, are vectors of unknown constants, $\mathbf{Q}^k, \dot{\mathbf{Q}}^k \in \mathbb{R}^{d \times r_k}$ are known time-varying matrices. Note that in this paper, we do not require the disturbance acting on the system $\mathbf{n}(t) = \mathbf{Q}^k(t)\mathbf{d}^k$ to have vanishing derivative $\lim_{t \rightarrow +\infty} \dot{\mathbf{n}}(t) = \mathbf{0}$ as in^{26,33}.

The disturbance observer is designed as follows

$$\dot{\mathbf{z}}^k = -(\mathbf{Q}^k)^\top (\mathbf{Q}^k(\mathbf{z}^k + (\mathbf{Q}^k)^\top \mathbf{x}^k) + \mathbf{x}^{k+1}) - \dot{\mathbf{Q}}_k^\top \mathbf{x}^k, \quad (11a)$$

$$\hat{\mathbf{d}}^k = \mathbf{z}^k + (\mathbf{Q}^k)^\top \mathbf{x}^k, \quad (11b)$$

for $k = 1, \dots, p-1$. The main result of this subsection is summarized in the following theorem.

Theorem 3. Suppose that $\mathbf{Q}^k(t)$, $k = 1, \dots, p$, are persistently exciting. Then, the following statements hold for the system (10) with the disturbance observer (11).

(i) Let $\mathbf{e}_d^k = \hat{\mathbf{d}}^k - \mathbf{d}^k$, then $\mathbf{e}_d^k = \mathbf{0}_{r_k}$ are globally exponentially stable.

(ii) Let the control input be given as

$$\mathbf{u} = -\mathbf{s} - \beta \text{sgn}(\mathbf{s}) - \sum_{k=1}^{p-1} \gamma_k \mathbf{x}^{k+1}, \quad (12a)$$

$$\mathbf{s} = \sum_{k=1}^{p-1} \gamma_{k+1} (\mathbf{x}^{k+1} + \mathbf{Q}^k \hat{\mathbf{d}}^k) + \gamma_1 \mathbf{x}^1, \quad (12b)$$

$$\dot{\beta} = \gamma \|\mathbf{s}\|_1, \beta(0) > 0, \quad (12c)$$

with $\gamma_p = 1$, $\gamma > 0$, and $\gamma_k > 0, k = 2, \dots, p-1$ are chosen so that the polynomial $P(s) = s^{p-1} + k_{p-1}s^{p-2} + \dots + k_2s + \gamma$ is Hurwitz. Then, it holds that $\lim_{t \rightarrow +\infty} \mathbf{s}(t) = \mathbf{0}_d$, $\lim_{t \rightarrow +\infty} \mathbf{x}^1(t) = \mathbf{0}_d$, and $\lim_{t \rightarrow +\infty} (\mathbf{x}^k(t) + \mathbf{Q}_{k-1} \mathbf{d}^{k-1}) = \mathbf{0}_d, \forall k = 2, \dots, p$.

Proof. Please see Appendix 6. □

We remark here that in the proof of Thm. 3, the LaSalle-Yoshizawa corollary^{38, Cor. 2} can also be used to derive the conclusion that $\lim_{t \rightarrow +\infty} \|\mathbf{s}(t)\| = 0$.

4 | DISTURBANCE OBSERVER BASED MATRIX-WEIGHTED CONSENSUS ALGORITHMS

In this section, we design disturbance-observer based matrix-weighted consensus algorithms under different assumptions on the agent's model. As practical systems are usually simplified by low-order systems, the multi-agent systems of single- and double integrators with disturbances will be firstly considered. Then, the disturbance-observer based consensus algorithm for a network of higher-order integrator agents with mismatched disturbances will be provided.

4.1 | Single integrator agents

Consider a matrix weighted network of single-integrators perturbed by disturbances. The disturbance on each agent $i = 1, \dots, n$, is modeled by $\mathbf{Q}_i(t)\mathbf{d}_i$, where $\mathbf{Q}_i(t) \in \mathbb{R}^{d \times r}$ is a time-varying matrix known by agent i and $\mathbf{d}_i \in \mathbb{R}^r$ is an unknown constant vector.[§] The agent's state vector is governed by the following equation

$$\dot{\mathbf{x}}_i = - \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j) + \mathbf{w}_i + \mathbf{Q}_i \mathbf{d}_i, \quad (13)$$

where $i = 1, \dots, n$, $\mathbf{u}_{ic} = - \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j)$ is the consensus algorithm, and \mathbf{w}_i is an additional input to compensate the disturbance. If the networks are disturbance-free, $\mathbf{Q}_i \mathbf{d}_i = \mathbf{0}_d, \forall i = 1, \dots, n$, the component \mathbf{u}_{ic} drives the agents to the average vector $\mathbf{x}^* = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i(0) \in \ker(\mathbf{L})$ ⁵.

[§] For simplicity, we assume that all disturbance vectors have the same dimension. This assumption is reasonable because if $r = \max_{i=1, \dots, n} r_i$, and for some $i \in \{1, \dots, n\}, r_i < r$, we can set $\mathbf{d}'_i = \text{vec}(\mathbf{d}_i, \mathbf{0}_{r-r_i})$ and $\mathbf{Q}'_i = [\mathbf{Q}_i \mathbf{0}_{d \times (n-r_i)}]$.

Based on the disturbance observer presented in Section 3, the disturbance observer for each agent is designed as follows

$$\dot{\mathbf{z}}_i = -\gamma \mathbf{Q}_i^\top \left(\mathbf{Q}_i \hat{\mathbf{d}}_i - \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij} (\mathbf{x}_i - \mathbf{x}_j) + \mathbf{w}_i \right) - \gamma \dot{\mathbf{Q}}_i^\top \mathbf{x}_i \quad (14a)$$

$$\hat{\mathbf{d}}_i = \mathbf{z}_i + \gamma \mathbf{Q}_i^\top \mathbf{x}_i, \quad i = 1, \dots, n, \quad (14b)$$

where $\gamma > 0$ is a control gain. It is clear that the disturbance observer (14) is decentralized in the sense that it only uses the weighted relative measurements $\mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j)$, $\forall j \in \mathcal{N}_i$, and the local variables \mathbf{x}_i .[¶]

Defining the vectors $\mathbf{w} = \text{vec}(\mathbf{w}_1, \dots, \mathbf{w}_n)$, $\mathbf{d} = \text{vec}(\mathbf{d}_1, \dots, \mathbf{d}_n)$, $\mathbf{z} = \text{vec}(\mathbf{z}_1, \dots, \mathbf{z}_n)$, $\hat{\mathbf{d}} = \text{vec}(\hat{\mathbf{d}}_1, \dots, \hat{\mathbf{d}}_n)$, and the matrix $\mathbf{Q} = \text{blkdiag}(\mathbf{Q}_1, \dots, \mathbf{Q}_n)$, the n -agent system (14) can be expressed in the matrix form

$$\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x} + \mathbf{w} + \mathbf{Q}\mathbf{d}, \quad (15a)$$

$$\dot{\mathbf{z}} = -\gamma \mathbf{Q}^\top \left(\mathbf{Q}\hat{\mathbf{d}} + (-\mathbf{L}\mathbf{x} + \mathbf{w}) \right) - \gamma \dot{\mathbf{Q}}^\top \mathbf{x}, \quad (15b)$$

$$\hat{\mathbf{d}} = \mathbf{z} + \gamma \mathbf{Q}^\top \mathbf{x}, \quad (15c)$$

and prove the following theorem.

Theorem 4. Let Assumption 1 hold. Consider the system (15) with the disturbance compensation law $\mathbf{w} = -\mathbf{Q}\hat{\mathbf{d}}$, then

- (i) $\mathbf{e}_d = \hat{\mathbf{d}} - \mathbf{d}$ is globally uniformly stable.
- (ii) In addition, if $\mathbf{Q}_i(t)$, $i = 1, \dots, n$ are persistently exciting, then $\mathbf{e}_d = \mathbf{0}$ is globally exponentially stable, and $\mathbf{x}(t)$ converges to a point $\mathbf{x}^* \in \ker(\mathbf{L})$, as $t \rightarrow \infty$ exponentially fast.

Proof. (i) From (15), it follows that

$$\dot{\mathbf{e}}_d = \dot{\mathbf{z}} + \gamma \mathbf{Q}^\top \dot{\mathbf{x}} + \gamma \dot{\mathbf{Q}}^\top \mathbf{x} = -\gamma \mathbf{Q}^\top \mathbf{Q} \mathbf{e}_d,$$

which implies that $\mathbf{e}_d = \mathbf{0}_{dn}$ is globally uniformly stable.

(ii) If the persistently exciting condition is satisfied for each \mathbf{Q}_i , then \mathbf{Q} is also persistently exciting. It follows that $\mathbf{e}_d = \mathbf{0}_{dr}$ is globally exponentially stable. The matrix-weighted consensus system (15a) can now be written as

$$\dot{\mathbf{x}} = -\mathbf{L}\mathbf{x} - \mathbf{Q}\mathbf{e}_d. \quad (16)$$

Expressing $\mathbf{L} = \mathbf{V} \text{diag}(\mathbf{0}_{l \times l}, \lambda_{l+1}, \dots, \lambda_{dn}) \mathbf{V}^\top$, for any eigenvalue-eigenvector pair $(\lambda_k, \mathbf{v}_k)$, $k = 1, \dots, dn$, by letting $y_k = \mathbf{v}_k^\top \mathbf{x}$, we have

$$\dot{y}_k = -\mathbf{v}_k^\top \mathbf{Q} \mathbf{e}_d, \quad k = 1, \dots, l, \quad (17a)$$

$$\dot{y}_k = -\lambda_k y_k - \mathbf{v}_k^\top \mathbf{Q} \mathbf{e}_d, \quad k = l+1, \dots, dn, \quad (17b)$$

As $\mathbf{e}_d \rightarrow \mathbf{0}_{dn}$ exponentially fast and $\mathbf{Q}(t)$ is bounded,[#] it follows that $\int_0^{+\infty} \mathbf{v}_k^\top \mathbf{Q}(\tau) \mathbf{e}_d(\tau) d\tau$ exists and is finite. Thus, $y_k(t) = y_k(0) - \int_0^{+\infty} \mathbf{v}_k^\top \mathbf{Q}(\tau) \mathbf{e}_d(\tau) d\tau$, $k = 1, \dots, l$, as $t \rightarrow +\infty$. In addition, as $\lambda_k > 0$, by linear system theory, each system (17b) has $y_k(t) \rightarrow 0$, $\forall k = l+1, \dots, dn$. Since y_k , $k = l+1, \dots, dn$, are state variables corresponding to the disagreement space $\ker(\mathbf{L})^\perp$, we conclude that $\mathbf{x}(t) \rightarrow \mathbf{x}^* \in \ker(\mathbf{L})$, as $t \rightarrow +\infty$. \square

Remark 2. Assume that each agent can sense only the relative states $\mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j)$ and exchange estimate variables with neighboring agents $j \in \mathcal{N}_i$. In this case, suppose that the disturbance on each agent is only modeled by a constant vector $\mathbf{d}_i \in \mathbb{R}^d$. Then, we can decompose $\mathbf{d} = \text{vec}(\mathbf{d}_1, \dots, \mathbf{d}_n) = \mathbf{d}^\parallel + \mathbf{d}^\perp$, where $\mathbf{d}^\parallel \in \ker(\mathbf{L})$ and $\mathbf{d}^\perp \perp \ker(\mathbf{L})$. Due to the restriction of the available information, only the perpendicular part, \mathbf{d}^\perp , of the disturbance can be estimated. The disturbance observer (14) can

[¶] Note that the disturbance observer (14) does not require all agents to have a common coordinate system but the axes of their local coordinate systems must be aligned.

[#] The boundedness of $\mathbf{Q}(t)$ is implied from the persistently exciting condition (3).

be redesigned as follows

$$\dot{\mathbf{z}}_i = -\gamma \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij} \left((\mathbf{z}_i - \mathbf{z}_j) + \gamma(\mathbf{x}_i - \mathbf{x}_j) \right) - \gamma \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij} \left(\left(-\sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j) + \mathbf{w}_i \right) - \left(-\sum_{k \in \mathcal{N}_j} \mathbf{A}_{jk}(\mathbf{x}_j - \mathbf{x}_k) + \mathbf{w}_j \right) \right), \quad (18a)$$

$$\hat{\mathbf{d}}_i = \mathbf{z}_i + \gamma \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i - \mathbf{x}_j), \quad i = 1, \dots, n \quad (18b)$$

The disturbance observer (18) can be expressed in a matrix form as follows

$$\dot{\hat{\mathbf{x}}} = -\mathbf{L}\mathbf{x} + \mathbf{w} + \mathbf{d}, \quad (19a)$$

$$\dot{\mathbf{z}} = -\gamma\mathbf{L}(\hat{\mathbf{x}} + \gamma\mathbf{L}\mathbf{x}) - \gamma\mathbf{L}(-\mathbf{L}\mathbf{x} + \mathbf{w}), \quad (19b)$$

$$\hat{\mathbf{d}} = \mathbf{z} + \gamma\mathbf{L}\mathbf{x}. \quad (19c)$$

We have $\dot{\hat{\mathbf{d}}} = -\gamma\mathbf{L}(\mathbf{z} + \gamma\mathbf{L}\mathbf{x}) - \gamma\mathbf{L}(-\mathbf{L}\mathbf{x} + \mathbf{w}) + \gamma\mathbf{L}(-\mathbf{L}\mathbf{x} + \mathbf{w} + \mathbf{d}) = -\gamma\mathbf{L}(\hat{\mathbf{d}} - \mathbf{d})$, or $\dot{\mathbf{e}}_d = -\gamma\mathbf{L}\mathbf{e}_d$. Thus, if $\mathbf{d} \perp \ker(\mathbf{L})$ and the auxiliary variables $\hat{\mathbf{x}}(0)$ are selected so that $\hat{\mathbf{d}}(0) = \mathbf{0}_{dn}$, then $\hat{\mathbf{d}}(t) \rightarrow \mathbf{d}$ exponentially fast and the input $\mathbf{w} = -\hat{\mathbf{d}}$ eventually compensates the unknown disturbance.

Note that in (19), 2-hop information (information from a neighbor of a neighbor) is needed for computing the input of each agent.

4.2 | Double integrator agents

In this subsection, we consider a matrix-weighted consensus network of double integrators with disturbances. First, we consider a disturbance observer based matrix-weighted consensus for agents with constant matched disturbances. Then, a corresponding observer-based matrix-weighted consensus for double integrators with mismatched disturbance will be proposed and analysed.

4.2.1 | Matched disturbance

Consider the matrix-weighted consensus system of double-integrators perturbed with a constant matched disturbance:

$$\dot{\mathbf{x}}_i^1 = \mathbf{x}_i^2, \quad (20a)$$

$$\dot{\mathbf{x}}_i^2 = -\sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i^1 - \mathbf{x}_j^1) - \mathbf{x}_i^2 + \mathbf{w}_i + \mathbf{Q}_i \mathbf{d}_i, \quad (20b)$$

where $\mathbf{x}_i^1, \mathbf{x}_i^2, \mathbf{Q}_i \mathbf{d}_i, \mathbf{w}_i \in \mathbb{R}^d$ are respectively the position, the velocity, the disturbance and the disturbance compensation input of agent i . The following disturbance observer is proposed for each agent

$$\dot{\mathbf{z}}_i = -\gamma \mathbf{Q}_i^\top \left(-\sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i^1 - \mathbf{x}_j^1) - \mathbf{x}_i^2 + \mathbf{w}_i - \mathbf{Q}_i(\mathbf{z}_i + \gamma \mathbf{Q}_i^\top \mathbf{x}_i^2) \right) - \gamma \dot{\mathbf{Q}}_i^\top \mathbf{x}_i^2, \quad (21a)$$

$$\hat{\mathbf{d}}_i = \mathbf{z}_i + \gamma \mathbf{Q}_i^\top \mathbf{x}_i^2, \quad i = 1, \dots, n, \quad (21b)$$

where $\mathbf{z}_i \in \mathbb{R}^d$ is an auxiliary variable, $\gamma > 0$ is a positive gain, and $\hat{\mathbf{d}}_i$ is an estimate of the disturbance. Let $\mathbf{x}^k = \text{vec}(\mathbf{x}_1^k, \dots, \mathbf{x}_n^k)$, $k = 1, 2$, $\mathbf{d} = \text{vec}(\mathbf{d}_1, \dots, \mathbf{d}_n)$, $\hat{\mathbf{d}} = \text{vec}(\hat{\mathbf{d}}_1, \dots, \hat{\mathbf{d}}_n)$, $\mathbf{z} = \text{vec}(\mathbf{z}_1, \dots, \mathbf{z}_n)$, and $\mathbf{e}_d = \hat{\mathbf{d}} - \mathbf{d}$, we have the following theorem.

Theorem 5. Suppose that Assumption 1 holds and the matrix \mathbf{Q} satisfies the persistently excitation condition (3). The following statements hold for the matrix-weighted consensus system (20) with the disturbance observer (21) and the disturbance compensation input $\mathbf{w}_i = -\mathbf{Q}_i \hat{\mathbf{d}}_i$.

- (i) $\mathbf{e}_d = \mathbf{0}_{dn}$ is globally exponentially stable.
- (ii) $\lim_{t \rightarrow +\infty} \mathbf{x}^1 = \mathbf{x}^{1*} \in \ker(\mathbf{L})$ and $\lim_{t \rightarrow +\infty} \mathbf{x}^2 = \mathbf{0}_{dn}$.

Proof. (i) Under the control law $\mathbf{w} = -\mathbf{Q}\hat{\mathbf{d}}$, we can rewrite the system (20) in the following matrix form

$$\dot{\mathbf{x}}^1 = \mathbf{x}^2, \quad (22a)$$

$$\dot{\mathbf{x}}^2 = -\mathbf{L}\mathbf{x}^1 - \mathbf{x}^2 - \mathbf{Q}\mathbf{e}_d, \quad (22b)$$

$$\dot{\mathbf{z}} = -\gamma\mathbf{Q}^\top(-\mathbf{L}\mathbf{x}^1 - \mathbf{x}^2 + \mathbf{w} + \mathbf{Q}\hat{\mathbf{d}}) - \gamma\dot{\mathbf{Q}}^\top\mathbf{x}^2 \quad (22c)$$

$$\dot{\hat{\mathbf{d}}} = \mathbf{z} + \gamma\mathbf{Q}^\top\mathbf{x}^2. \quad (22d)$$

Then, the estimation error $\mathbf{e}_d = \hat{\mathbf{d}} - \mathbf{d}$ evolves according to the equation

$$\dot{\mathbf{e}}_d = \dot{\mathbf{z}} + \gamma\mathbf{Q}^\top\dot{\mathbf{x}}^2 + \gamma\dot{\mathbf{Q}}^\top\mathbf{x}^2 = -\gamma\mathbf{Q}^\top\mathbf{Q}\mathbf{e}_d.$$

As $\mathbf{Q}(t)$ is persistently exciting, based on Lemma 2, we conclude that $\mathbf{e}_d = \mathbf{0}_{dn}$ is globally exponentially stable.

(ii) Now, $\mathbf{Q}\mathbf{e}_d$ can be considered as a vanishing disturbance acting on the system

$$\begin{bmatrix} \dot{\mathbf{x}}^1 \\ \dot{\mathbf{x}}^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0}_{dn \times dn} & \mathbf{I}_{dn} \\ -\mathbf{L} & -\mathbf{I}_{dn} \end{bmatrix}}_{:=\mathbf{M}} \begin{bmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \end{bmatrix} - \begin{bmatrix} \mathbf{0}_{dn} \\ \mathbf{I}_{dn} \end{bmatrix} \mathbf{Q}\mathbf{e}_d. \quad (23)$$

The matrix \mathbf{M} has l zero eigenvalues ($d \leq l \leq dn$) and $2dn - l$ eigenvalues with negative real parts, and we can write $\mathbf{M} = \mathbf{R}\mathbf{b}\mathbf{l}\mathbf{k}\mathbf{d}\mathbf{i}\mathbf{a}\mathbf{g}(\mathbf{0}_{l \times l}, \mathbf{J}_M)\mathbf{R}^{-1}$, where \mathbf{J}_M contains all Jordan blocks corresponding to nonzero eigenvalues of \mathbf{M} . The left and right eigenvectors corresponding to l zero eigenvalues of \mathbf{M} are $\mathbf{p}_k = \text{vec}(\mathbf{v}_k, \mathbf{v}_k)$, and $\mathbf{r}_k = \text{vec}(\mathbf{v}_k, \mathbf{0}_{dn})$, $k = 1, \dots, l$, where \mathbf{v}_k , $k = 1, \dots, l$, are eigenvectors corresponding to the zero eigenvalues of \mathbf{L} . Let $\mathbf{y} = \mathbf{R}^{-1}\text{vec}(\mathbf{x}^1, \mathbf{x}^2)$, then $\dot{\mathbf{y}}_k = \mathbf{p}_k^\top \text{vec}(\dot{\mathbf{x}}^1, \dot{\mathbf{x}}^2) = -\mathbf{v}_k^\top \mathbf{Q}\mathbf{e}_d$, $k = 1, \dots, l$. Since $\mathbf{e}_d \rightarrow \mathbf{0}_{dn}$ exponentially fast and $\mathbf{Q}(t)$ is bounded, $\int_0^{+\infty} \mathbf{v}_k^\top \mathbf{Q}(\tau)\mathbf{e}_d(\tau)d\tau$ exists and is finite. Therefore, as $t \rightarrow +\infty$, $y_k(t) \rightarrow y_k^* = y_k(0) - \int_0^{+\infty} \mathbf{v}_k^\top \mathbf{Q}(\tau)\mathbf{e}_d(\tau)d\tau$, where $y_k(0) = \mathbf{v}_k^\top(\mathbf{x}^1(0) + \mathbf{x}^2(0))$, $k = 1, \dots, l$.

For $k = l+1, \dots, 2dn$, we can write

$$\dot{\mathbf{y}}_{[l+1:2dn]} = \mathbf{J}_M \mathbf{y}_{[l+1:2dn]} - \mathbf{e}_d',$$

where $\mathbf{y}_{[l+1:2dn]} = [y_{l+1}, \dots, y_{2dn}]^\top$, \mathbf{J}_M is Hurwitz, and $\mathbf{e}_d' \rightarrow \mathbf{0}_{2dn-l}$ exponentially fast. By using Lemma 1, $\mathbf{y}_{[l+1:2dn]} \rightarrow \mathbf{0}_{2dn-l}$ as $t \rightarrow +\infty$. In summary, denoting $\mathbf{y}_{[1:l]}^* = [y_1^*, \dots, y_l^*]^\top$, then $\lim_{t \rightarrow +\infty} \text{vec}(\mathbf{x}^1, \mathbf{x}^2) = \lim_{t \rightarrow +\infty} \mathbf{R}\mathbf{y} = \begin{bmatrix} [\mathbf{v}_1, \dots, \mathbf{v}_l]\mathbf{y}_{[1:l]}^* \\ \mathbf{0}_{dn} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^l y_k^* \mathbf{v}_k \\ \mathbf{0}_{dn} \end{bmatrix}.$ \square

4.2.2 | Mismatched disturbance

Now, we consider the matrix-weighted consensus problem of double-integrator agents with mismatched disturbances. Suppose that a time-varying disturbance $\mathbf{Q}_i(t)\mathbf{d}_i$, where $\mathbf{d}_i \in \mathbb{R}^r$ is an unknown constant and $\mathbf{Q}_i(t)$ is a time-varying known matrix, acts on the velocity of each agent, i.e.,

$$\dot{\mathbf{x}}_i^1 = \mathbf{x}_i^2 + \mathbf{Q}_i\mathbf{d}_i, \quad (24a)$$

$$\dot{\mathbf{x}}_i^2 = \mathbf{w}_i, \quad (24b)$$

where $i = 1, \dots, n$. In this scenario, the disturbance is mismatched with the control input \mathbf{w}_i .

The following disturbance observer for (24) is proposed

$$\dot{\mathbf{z}}_i = -\mathbf{Q}_i^\top(\mathbf{Q}_i(\mathbf{z}_i + \mathbf{Q}_i^\top\mathbf{x}_i^1) + \mathbf{x}_i^2) - \dot{\mathbf{Q}}_i^\top\mathbf{x}_i^1, \quad (25a)$$

$$\hat{\mathbf{d}}_i = \mathbf{z}_i + \mathbf{Q}_i^\top\mathbf{x}_i^1, \quad i = 1, \dots, n. \quad (25b)$$

Let $\mathbf{s}_i = \mathbf{x}_i^2 + \gamma \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i^1 - \mathbf{x}_j^1) + \mathbf{Q}_i \hat{\mathbf{d}}_i$, $i = 1, \dots, n$, and $\gamma > 0$ is a control gain. Consider the disturbance compensation law³⁹

$$\mathbf{w}_i = -\mathbf{s}_i - \gamma \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i^2 - \mathbf{x}_j^2) - \beta_i(t) \frac{\mathbf{s}_i}{\|\mathbf{s}_i\| + \mu_i(t)} - \dot{\mathbf{Q}}_i \hat{\mathbf{d}}_i, \quad (26a)$$

$$\dot{\beta}_i(t) = \gamma_i \frac{\|\mathbf{s}_i\|^2}{\|\mathbf{s}_i\| + \mu_i(t)}, \quad \beta_i(0) > 0, \quad i = 1, \dots, n, \quad (26b)$$

where $\mu_i(t)$ is chosen such that $\mu_i(t) > 0, \forall t \geq 0$, and $\int_0^{+\infty} \mu_i(\tau) d\tau < \infty$.^{||} We prove the following theorem.

Theorem 6. Suppose that Assumption 1 holds and $\mathbf{Q}_i(t)$, $i = 1, \dots, n$, are persistently exciting. Consider the matrix-weighted consensus system (24) with the disturbance observer (25) and the disturbance compensation law (26). The following statements hold:

- (i) The disturbance estimation error $\mathbf{e}_d = \hat{\mathbf{d}} - \mathbf{d}$ globally exponentially converges to $\mathbf{0}_{dn}$,
- (ii) $\lim_{t \rightarrow +\infty} \mathbf{s}(t) \rightarrow \mathbf{0}_{dn}$, and $\lim_{t \rightarrow +\infty} \beta_i(t)$, $i = 1, \dots, n$, exist and are finite,
- (iii) $\lim_{t \rightarrow +\infty} \mathbf{x}^1(t) = \mathbf{x}^{*1} \in \ker(\mathbf{L})$ for some constant vector \mathbf{x}^{*1} , and $\lim_{t \rightarrow +\infty} \mathbf{x}^2(t) = -\mathbf{d}$.

Proof. (i) By expressing the system in a matrix form, we have

$$\dot{\mathbf{x}}^1 = \mathbf{x}^2 + \mathbf{Q}\mathbf{d}, \quad (27a)$$

$$\dot{\mathbf{x}}^2 = \mathbf{w}, \quad (27b)$$

$$\dot{\mathbf{z}} = -\mathbf{Q}^\top(\mathbf{Q}\hat{\mathbf{d}} + \mathbf{x}^2) - \dot{\mathbf{Q}}^\top \mathbf{x}^1, \quad (27c)$$

$$\hat{\mathbf{d}} = \mathbf{z} + \mathbf{Q}^\top \mathbf{x}^1, \quad (27d)$$

$$\mathbf{s} = \mathbf{x}^2 + \gamma \mathbf{L} \mathbf{x}^1 + \mathbf{Q}\hat{\mathbf{d}}, \quad (27e)$$

$$\dot{\mathbf{s}} = \mathbf{w} + \gamma \mathbf{L} \mathbf{x}^2 + \gamma \mathbf{L} \mathbf{Q} \mathbf{d} + \dot{\mathbf{Q}} \hat{\mathbf{d}} + \mathbf{Q} \dot{\hat{\mathbf{d}}}, \quad (27f)$$

$$\mathbf{w} = -\mathbf{s} - \gamma \mathbf{L} \mathbf{x}^2 - \dot{\mathbf{Q}} \hat{\mathbf{d}} - \left(\text{diag} \left(\frac{\beta_i}{\|\mathbf{s}_i\| + \mu_i(t)} \right) \otimes \mathbf{I}_d \right) \mathbf{s}, \quad (27g)$$

$$\dot{\beta} = \text{blkdiag} \left(\frac{\gamma_i \mathbf{s}_i^\top}{\|\mathbf{s}_i\| + \mu_i(t)} \right) \mathbf{s}, \quad (27h)$$

where $\mathbf{s} = \text{vec}(\mathbf{s}_1, \dots, \mathbf{s}_n)$, $\beta = [\beta_1, \dots, \beta_n]^\top$, and \otimes denotes the Kronecker product. We have $\dot{\mathbf{e}}_d = \dot{\hat{\mathbf{d}}} - \dot{\mathbf{d}} = \dot{\mathbf{z}} + \dot{\mathbf{Q}} \mathbf{x}^1 + \mathbf{Q}^\top \dot{\mathbf{x}}^1 = -\mathbf{Q}^\top \mathbf{Q} \mathbf{e}_d$. It follows from Lemma 2 that $\mathbf{e}_d \rightarrow \mathbf{0}_{dn}$ exponentially fast.

(ii) Consider the Lyapunov function $V = \frac{1}{2} \mathbf{s}^\top \mathbf{s} + \frac{1}{2} \sum_{i=1}^n \frac{1}{\gamma_i} (\beta_i - \beta_{\max})^2$, where $\beta_{\max} > \sup_{t \geq 0} \{ \|\mathbf{Q} \mathbf{Q}^\top \mathbf{Q} \mathbf{e}_d\| + \gamma \|\mathbf{L} \mathbf{Q} \mathbf{d}\| \}$. Clearly, V is positive definite with regard to $\text{vec}(\mathbf{s}, \beta)$ and being bounded by two class- \mathcal{K}_∞ functions $\frac{1}{2} \mathbf{s}^\top \mathbf{s} + \frac{1}{2 \max_i \gamma_i} \|\beta - \beta_{\max} \mathbf{1}_n\|^2$ and $\frac{1}{2} \mathbf{s}^\top \mathbf{s} + \frac{1}{2 \min_i \gamma_i} \|\beta - \beta_{\max} \mathbf{1}_n\|^2$. We have,

$$\begin{aligned} \dot{V} &= -\mathbf{s}^\top \left(\text{diag} \left(\frac{\beta_i}{\|\mathbf{s}_i\| + \mu_i(t)} \right) \otimes \mathbf{I}_d \right) \mathbf{s} - \mathbf{s}^\top \mathbf{s} + \mathbf{s}^\top (\gamma \mathbf{L} \mathbf{Q} \mathbf{d} - \mathbf{Q} \mathbf{Q}^\top \mathbf{Q} \mathbf{e}_d) + \sum_{i=1}^n (\beta_i - \beta_{\max}) \frac{\|\mathbf{s}_i\|^2}{\|\mathbf{s}_i\| + \mu_i(t)} \\ &\leq -\|\mathbf{s}\|^2 + \sum_{i=1}^n \frac{\beta_{\max} \mu_i(t) \|\mathbf{s}_i\|}{\|\mathbf{s}_i\| + \mu_i(t)} \\ &\leq -\|\mathbf{s}\|^2 + \beta_{\max} \sum_{i=1}^n \mu_i(t). \end{aligned} \quad (28)$$

It follows that $0 \leq V(t) + \int_0^t \|\mathbf{s}(\tau)\|^2 d\tau \leq V(0) + \beta_{\max} \sum_{i=1}^n \int_0^t \mu_i(\tau) d\tau$. Thus, $\lim_{t \rightarrow +\infty} V(t) < \infty$, $\int_0^{+\infty} \|\mathbf{s}(\tau)\|^2 d\tau < \infty$, and β_i are bounded. From (27f)–(27g), considering $\mathbf{L} \mathbf{Q} \mathbf{d} - \mathbf{Q} \mathbf{Q}^\top \mathbf{Q} \mathbf{e}_d$ as a bounded input to the system, it follows that \mathbf{s} is (globally) uniformly bounded. Thus, $\|\dot{\mathbf{s}}\| \leq (1 + \beta_{\max}) \|\mathbf{s}\| + \beta_{\max}$, or $\dot{\mathbf{s}}$ is uniformly bounded. By Barbalat's lemma, $\lim_{t \rightarrow +\infty} \|\mathbf{s}\| = 0$. Moreover, from Eqn. (26), $\beta_i(t)$ are continuous, non-decreasing, and upper bounded. Thus, $\exists \beta_i^* = \lim_{t \rightarrow +\infty} \beta_i(t)$, $i = 1, \dots, n$.

^{||} For example, we can choose $\mu_i(t) = \exp(-t)$.

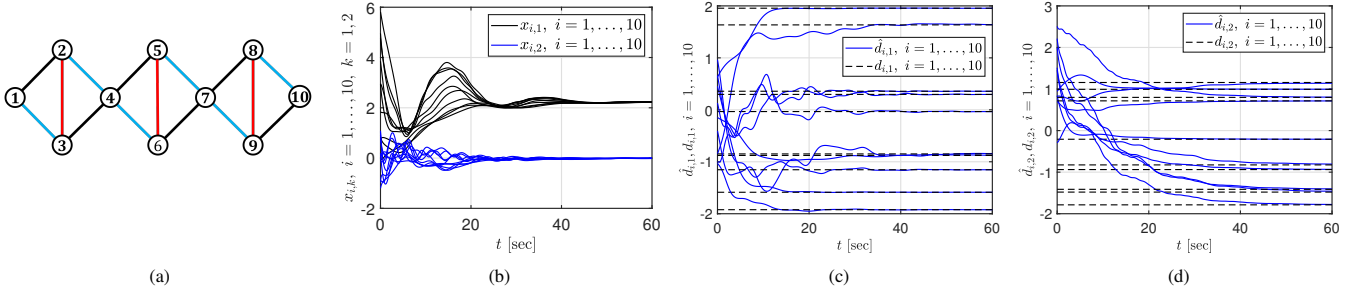


FIGURE 2 Simulation 1: (a) the matrix weighted graph \mathcal{G} ; (b) $\mathbf{x}_i(t)$; (c) and (d) $\hat{\mathbf{d}}_i(t)$ and \mathbf{d}_i , for $i = 1, \dots, 10$.

(iii) Now, we rewrite the system (27e) as

$$\dot{\mathbf{x}}^1 = -\gamma \mathbf{L} \mathbf{x}^1 - \mathbf{Q} \mathbf{e}_d + \mathbf{s}, \quad (29)$$

and consider $\boldsymbol{\eta} = -\mathbf{Q} \mathbf{e}_d + \mathbf{s}$ as a vanishing input to the nominal consensus system. Similar to Thm. 4 (ii), and from the fact that $\int_0^{+\infty} \|\mathbf{s}(\tau)\|^2 d\tau$ is finite, it follows that $\mathbf{x}^1(t) \rightarrow \mathbf{x}^* \in \ker(\mathbf{L})$, $\dot{\mathbf{x}}^1(t) \rightarrow \mathbf{0}_{dn}$, and thus $\lim_{t \rightarrow +\infty} (\mathbf{x}^2(t) + \mathbf{Q} \mathbf{d}) = \mathbf{0}_{dn}$. \square

Remark 3. We remark that the following adaptive sliding-mode based consensus law can be proposed instead of (26): $\mathbf{w}_i = -\mathbf{s}_i - \gamma \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i^1 - \mathbf{x}_j^1) - \beta_i(t) \text{sgn}(\mathbf{s}_i)$, $\dot{\beta}_i(t) = \gamma_i \|\mathbf{s}_i\|_1$. A similar stability analysis can be conducted as in the proof of Theorems 3 and 6. The consensus law (26) is selected since it reduces sudden changes in the control's direction caused by the discontinuity of the signum function.

4.3 | Higher-order integrator agents with mismatched disturbances

In this subsection, we generalize the approach in subsection 4.2.2 to the matrix weighted consensus problem of higher-order integrator systems with mismatched disturbances. The result is summarized in the following theorem.

Theorem 7. Let Assumption 1 hold. Consider the matrix-weighted consensus system of order $p \in \mathbb{N}$, $p \geq 3$, perturbed by some mismatched disturbances

$$\dot{\mathbf{x}}_i^k = \mathbf{x}_i^{k+1} + \mathbf{Q}_i^k \mathbf{d}_i^k, \quad k = 1, \dots, p-1, \quad (30a)$$

$$\dot{\mathbf{x}}_i^p = \mathbf{w}_i, \quad (30b)$$

where $i = 1, \dots, n$, and suppose that the matrices \mathbf{Q}_i^k , $k = 1, \dots, p-1$, are persistently exciting. Let the disturbance observer be designed as follows

$$\dot{\mathbf{z}}_i^k = -(\mathbf{Q}_i^k)^\top \mathbf{Q}_i^k \hat{\mathbf{d}}_i^k - (\mathbf{Q}_i^k)^\top \mathbf{x}_i^{k+1} - (\dot{\mathbf{Q}}_i^k)^\top \mathbf{x}_i^k, \quad (31a)$$

$$\hat{\mathbf{d}}_i^k = \mathbf{z}_i^k + (\mathbf{Q}_i^k)^\top \mathbf{x}_i^k, \quad (31b)$$

for $k = 1, \dots, p-1$, $i = 1, \dots, n$. Defining

$$\mathbf{s}_i = \mu_1 \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i^1 - \mathbf{x}_j^1) + \sum_{k=2}^p \mu_k \mathbf{x}_i^k + \sum_{k=1}^{p-1} \mu_{k+1} \mathbf{Q}_i^k \hat{\mathbf{d}}_i^k,$$

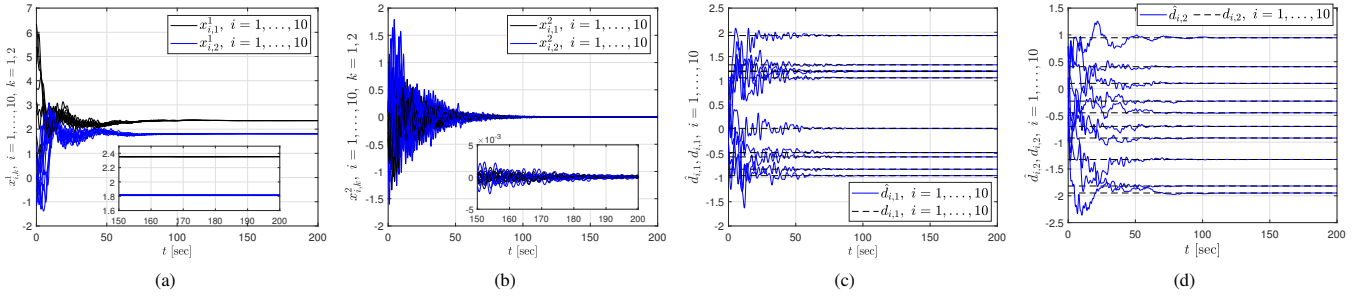


FIGURE 3 Simulation 2: (a) $\mathbf{x}_i^1(t)$; (b) $\mathbf{x}_i^2(t)$; (c) and (d) $\hat{\mathbf{d}}_i$ and \mathbf{d}_i , for $i = 1, \dots, 10$.

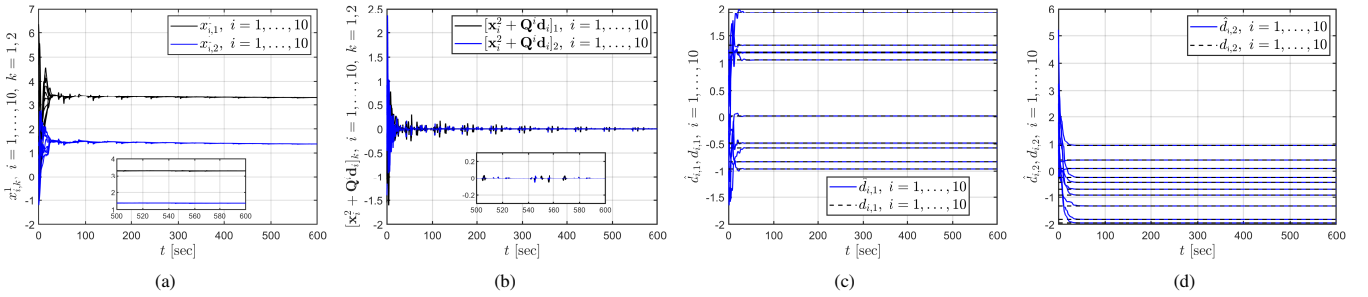


FIGURE 4 Simulation 3: (a) $\mathbf{x}_i^1(t)$; (b) $\mathbf{x}_i^2(t) + \mathbf{Q}_i(t)\mathbf{d}_i$; (c) and (d) $\hat{\mathbf{d}}_i$ and \mathbf{d}_i , for $i = 1, \dots, 10$.

where $i = 1, \dots, n$, $\mu_p = 1$, and $\mu_1, \dots, \mu_{p-1} > 0$ are chosen so that the following polynomials are Hurwitz

$$Q_{\lambda_r}(s) = \mu_1 \lambda_r + \sum_{q=1}^{p-2} \mu_{q+1} s^q + s^{p-1}, \quad (32a)$$

$$Q_0(s) = \sum_{q=1}^{p-2} \mu_{q+1} s^{q-1} + s^{p-2}, \quad (32b)$$

where λ_r , $r = l+1, \dots, dn$, are positive eigenvalues of the matrix-weighted Laplacian \mathbf{L} . Then, under the disturbance compensation input

$$\mathbf{w}_i = -\mu_1 \sum_{j \in \mathcal{N}_i} \mathbf{A}_{ij}(\mathbf{x}_i^2 - \mathbf{x}_j^2) - \sum_{k=2}^{p-1} \mu_k \mathbf{x}_i^{k+1} - \mathbf{s}_i - \beta_i(t) \frac{\mathbf{s}_i}{\|\mathbf{s}_i\| + \mu_i(t)} - \sum_{k=1}^{p-1} \mu_{k+1} \dot{\mathbf{Q}}_i^k \hat{\mathbf{d}}_i^k, \quad (33a)$$

$$\dot{\beta}_i = \gamma_i \frac{\|\mathbf{s}_i\|^2}{\|\mathbf{s}_i\| + \mu_i(t)}, \quad \beta_i(0) > 0, \quad i = 1, \dots, n, \quad (33b)$$

we have $\lim_{t \rightarrow +\infty} \mathbf{L}\mathbf{x}^1(t) = \mathbf{0}_{dn}$, and $\lim_{t \rightarrow +\infty} (\mathbf{x}^k(t) + \mathbf{Q}^{k-1} \mathbf{d}^{k-1}) = \mathbf{0}_{dn}$, $\forall k = 2, \dots, p$.

Proof. See Appendix 6. □

Note that for a network of p -th order integrator agents ($p \geq 3$), the proposed control law only guarantees that $\lim_{t \rightarrow +\infty} \mathbf{L}\mathbf{x}^1(t) = \mathbf{0}_{dn}$. To guarantee \mathbf{x}^1 to converge to some point $\mathbf{x}^{1*} \in \ker(\mathbf{L})$, a sufficient condition is $\|\mathbf{s}\|$ converges to $\mathbf{0}$ at an exponential rate. This can be achieved by a sliding-mode control law given that the upper bounds of $\|\mathbf{L}\|$, $\|\mathbf{d}^k\|$, and $\|\dot{\mathbf{Q}}_i^k\|$ are known.

5 | SIMULATION RESULTS

In this section, several simulations will be given to support the design and stability analysis in the previous sections. Consider

the graph \mathcal{G} as depicted in Fig. 2(a), where the (black, blue, red) edges have the corresponding matrix weights

$$\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ (p.s.d.)}, \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ (p.s.d.)}, \text{ and } \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (p.d.)}.$$

Thus, the matrix \mathbf{L} satisfies Assumption 1, i.e., $\text{rank}(\mathbf{L}) = 18$. In the simulations, we select the matrices $\mathbf{Q}_i = \begin{bmatrix} \sin(0.1it) & 0.2\exp(-it) \\ 0.5\cos(0.3t) & 0.4\cos(0.2it + 0.15) \end{bmatrix}$, $i = 1, \dots, 10$, the constant gain $\gamma = 1$, and randomly generate the constant vector \mathbf{d} and the initial states $\mathbf{x}(0)$.

5.1 | Disturbance observer-based matrix-weighted consensus of single-integrators

Let the agents be modeled by single integrators with disturbance. We conduct a simulation of the system (15) and obtain the results as depicted in Fig. 2 (b)–(d). The disturbance estimation errors become very small after about 15 seconds. The agents' states $\mathbf{x}_i(t)$ asymptotically converge to a common vector.

5.2 | Disturbance observer-based matrix-weighted consensus of double-integrators

5.2.1 | Matched disturbance

With the same interaction graph, the matrices \mathbf{Q}_i , and the disturbance vector, as in Simulation 1, we simulate the system (20)–(21). As can be seen from Fig. 3 (a)–(c), \mathbf{x}_i^1 asymptotically achieves a consensus, $\mathbf{x}_i^2 \rightarrow \mathbf{0}_{dn}$, and the disturbance estimates converge to the true value exponentially fast.

5.2.2 | Mismatched disturbance

Finally, we simulate the consensus system with mismatched disturbance (24), (25), (26) with $\mu(t) = \exp(-0.3t)$, $\gamma_i = 0.3$, $\forall i = 1, \dots, 10$. From Fig. 4, \mathbf{e}_d exponentially converges to $\mathbf{0}$ (\mathbf{e}_d is very small after 20 seconds), and \mathbf{x}_i^1 , $i = 1, \dots, 10$, eventually achieve a consensus and $\mathbf{x}_i^2(t)$ eventually eliminate the effects of the mismatched disturbances, i.e., $\mathbf{x}_i^2(t) + \mathbf{Q}_i \mathbf{d}_i \rightarrow \mathbf{0}_d$. The simulation results, thus, are consistent with Section 4.

6 | CONCLUSIONS

In this paper, we have firstly proposed disturbance observers for linear systems with disturbance modeled by a product of time-varying known matrices with an unknown parameter vector. Confining on single- and higher-order integrator systems, disturbance observer-based control laws were proposed in case that the disturbances are matched or mismatched with the control input.

Second, the new disturbance-observer based control method has been applied to the matrix-weighted consensus problem. We provide mathematical analysis to show that for networks of single- or higher-order integrators, the disturbances are estimated exponentially fast and the system asymptotically reaches the kernel of the matrix-weighted Laplacian, which may correspond to a consensus or a clustering configuration.

For further studies, many properties of matrix-weighted networks which do not exhibit in scalar weighted graphs have yet to be studied. These properties would provide new insights on multi-layer network as well as its applications. Another direction is applying the proposed disturbance observers to other multi-agent systems with disturbances such as formation control, network localization, distributed optimization,...

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APPENDIX

PROOF OF THEOREM 3

(i) We have $\dot{\mathbf{e}}_d^k = \dot{\mathbf{d}}^k = -(\mathbf{Q}^k)^\top \mathbf{Q}^k \mathbf{e}_d^k$. Since $(\mathbf{Q}^k)^\top \mathbf{Q}^k$ is symmetric positive semidefinite, it follows that \mathbf{e}_d^k , $k = 1, \dots, p-1$, is uniformly stable. Based on Lemma 2, as the persistently exciting condition (3) is satisfied, we conclude that $\mathbf{e}_d^k \rightarrow \mathbf{0}_{r_k}$, $k = 1, \dots, p-1$, at an exponential rate.

(ii) We have

$$\begin{aligned} \dot{\mathbf{s}} &= \mathbf{u} + \sum_{k=1}^{p-1} \gamma_k \mathbf{x}^{k+1} + \boldsymbol{\eta}(t) \\ &= -\mathbf{s} - \beta \operatorname{sgn}(\mathbf{s}) + \boldsymbol{\eta}(t), \end{aligned} \quad (1)$$

where $\boldsymbol{\eta}(t) = \sum_{k=1}^{p-1} \gamma_k \mathbf{Q}^k \mathbf{d}^k - \sum_{k=1}^{p-1} \gamma_{k+1} (\mathbf{Q}^k)^\top \mathbf{Q}^k \mathbf{e}_d^k$ is an unknown, bounded vector.

Consider the Lyapunov function $V = \frac{1}{2} \|\mathbf{s}\|^2 + \frac{1}{2\gamma} (\beta - \beta_{\max})^2$, where $\beta_{\max} > \sup_{t \geq 0} \|\boldsymbol{\eta}\|_\infty$. Then, V is continuous, positive definite and radially unbounded with regard to $\operatorname{vec}(\mathbf{s}, \beta - \beta_{\max})$. Since (1) has discontinuous right-hand-side, the solution of (1) is understood in the Filippov's sense⁴⁰. By using the formula $xK[\operatorname{sign}(x)] = |x|$, we have

$$\begin{aligned} \dot{V} &\in \text{a.e. } \dot{V} = \bigcap_{\boldsymbol{\xi} \in \partial V} \boldsymbol{\xi}^\top K[\dot{\mathbf{s}}] \\ &= \mathbf{s}^\top (-\mathbf{s} - \beta K[\operatorname{sgn}(\mathbf{s})] + \boldsymbol{\eta}) + (\beta - \beta_{\max}) \|\mathbf{s}\|_1 \\ &= -\|\mathbf{s}\|^2 - \beta \|\mathbf{s}\|_1 + \mathbf{s}^\top \boldsymbol{\eta} + (\beta - \beta_{\max}) \|\mathbf{s}\|_1 \\ &\leq -\|\mathbf{s}\|^2 - (\beta_{\max} - \|\boldsymbol{\eta}\|_\infty) \|\mathbf{s}\|_1 \leq 0. \end{aligned}$$

Thus, $\operatorname{vec}(\mathbf{s}, \beta - \beta_{\max})$ is uniformly bounded, and $0 \leq V(t) + \int_0^t \|\mathbf{s}(\tau)\|^2 d\tau \leq V(0)$. Thus, $\lim_{t \rightarrow +\infty} V(t)$ exists and is finite, and $\beta(t)$ is bounded. Since $\dot{\mathbf{s}}$ is bounded, \mathbf{s} is uniformly continuous. As $\boldsymbol{\eta}$ is also uniformly continuous, Barbalat's lemma can be invoked to conclude that $\lim_{t \rightarrow +\infty} \|\mathbf{s}\| = 0$. Also, as $\beta(t)$ is non-decreasing and being upper-bounded, it follows that $\lim_{t \rightarrow +\infty} \beta(t) = \beta^*$ exists and is finite.

Next, we rewrite Eqn. (12)(b) as follows

$$\begin{aligned} \mathbf{s} &= \sum_{k=1}^{p-1} \gamma_{k+1} \dot{\mathbf{x}}^k + \gamma_1 \mathbf{x}^1 - \sum_{k=1}^{p-1} \gamma_{k+1} \mathbf{Q}^k \mathbf{e}_d^k \\ \frac{d^{p-1} \mathbf{x}^1}{dt^{p-1}} &= -\gamma_1 \mathbf{x}^1 - \gamma_2 \frac{d\mathbf{x}^1}{dt} - \dots - \gamma_{p-1} \frac{d^{p-2} \mathbf{x}^1}{dt^{p-2}} + \boldsymbol{\zeta}, \end{aligned} \quad (2)$$

where $\boldsymbol{\zeta} = \mathbf{s} + \sum_{k=1}^{p-1} \gamma_{k+1} \mathbf{Q}^k \mathbf{e}_d^k$. Thus, we can consider $\boldsymbol{\zeta}$ as a bounded and vanishing input acting on the nominal continuous time-invariant linear system

$$\frac{d^{p-1} \mathbf{x}^1}{dt^{p-1}} = -\gamma_1 \mathbf{x}^1 - \gamma_2 \frac{d\mathbf{x}^1}{dt} - \dots - \gamma_{p-1} \frac{d^{p-2} \mathbf{x}^1}{dt^{p-2}},$$

with state variables $\mathbf{y}^k = \frac{d^k \mathbf{x}^1}{dt^k}$, $k = 0, \dots, p-1$, of which the characteristic equation $P(s)$ is Hurwitz. Thus, by Lemma 1, for system (2), we conclude that $\lim_{t \rightarrow +\infty} \mathbf{y}^k \rightarrow \mathbf{0}_d$, $k = 1, \dots, p-1$. Equivalently, $\lim_{t \rightarrow +\infty} \mathbf{x}^1 = \mathbf{0}_d$ and $\lim_{t \rightarrow +\infty} (\mathbf{x}^k + \mathbf{Q}_{k-1} \mathbf{d}^{k-1}) = \mathbf{0}_d$ for all $k = 2, \dots, p$. This completes the proof.

PROOF OF THEOREM 6

The proof consists of three parts. First, we prove that the disturbance observer makes $\mathbf{e}_d^k = \hat{\mathbf{d}}^k - \mathbf{d}^k$ globally exponentially converges to $\mathbf{0}_{dn}$. Indeed, this statement follows immediately from

$$\begin{aligned}\dot{\mathbf{e}}_{di}^k &= \dot{\mathbf{z}}_i^k + (\dot{\mathbf{Q}}_i^k)^\top \mathbf{x}_i^k + (\mathbf{Q}_i^k)^\top \dot{\mathbf{x}}^k \\ &= -(\mathbf{Q}_i^k)^\top \mathbf{Q}_i^k \mathbf{e}_{di}^k,\end{aligned}$$

the persistent excitation assumption on \mathbf{Q}_i^k , $k = 1, \dots, p-1$, $i = 1, \dots, n$, and Lemma 2.

Second, denoting $\mathbf{Q}^k = \text{blkdiag}(\mathbf{Q}_1^k, \dots, \mathbf{Q}_n^k)$ and $\mathbf{d}^k = \text{vec}(\mathbf{d}^1, \dots, \mathbf{d}^{p-1})$, we have

$$\begin{aligned}\mathbf{s} &= \mathbf{x}^p + \sum_{k=2}^{p-1} \mu_k \mathbf{x}^k + \mu_1 \mathbf{L} \mathbf{x}^1 + \sum_{k=2}^{p-1} \mu_{k+1} \mathbf{Q}_i^k \hat{\mathbf{d}}^k \\ &= \frac{d^{p-1}}{dt^{p-1}} \mathbf{x}^1 - \mathbf{Q}^{p-1} \mathbf{d}^{p-1} + \sum_{k=1}^{p-2} \mu_{k+1} \left(\frac{d^k}{dt^k} \mathbf{x}^1 - \mathbf{Q}^k \mathbf{d}^k \right) + \mu_1 \mathbf{L} \mathbf{x}^1 + \mathbf{Q}^{p-1} \hat{\mathbf{d}}^{p-1} + \sum_{k=1}^{p-2} \mu_{k+1} \mathbf{Q}^k \hat{\mathbf{d}}^k \\ &= \frac{d^{p-1}}{dt^{p-1}} \mathbf{x}^1 + \sum_{k=1}^{p-2} \mu_{k+1} \frac{d^k \mathbf{x}^1}{dt^k} + \mu_1 \mathbf{L} \mathbf{x}^1 + \sum_{k=1}^{p-2} \mu_{k+1} \mathbf{Q}^k \mathbf{e}_d^k + \mathbf{Q}^{p-1} \mathbf{e}_d^{p-1}.\end{aligned}\quad (3)$$

$$\begin{aligned}\dot{\mathbf{s}} &= \dot{\mathbf{w}} + \sum_{k=2}^{p-1} \mu_k (\mathbf{x}^{k+1} + \mathbf{Q}^k \mathbf{d}^k) + \mu_1 \mathbf{L} (\mathbf{x}^2 + \mathbf{Q}^1 \mathbf{d}^1) + \sum_{k=1}^{p-1} \mu_{k+1} \dot{\mathbf{Q}}^k \hat{\mathbf{d}}^k + \sum_{k=1}^{p-1} \mu_{k+1} \mathbf{Q}^k (\mathbf{Q}^k)^\top \mathbf{Q}^k \dot{\mathbf{e}}_d^k \\ &= \dot{\mathbf{w}} + \mu_1 \mathbf{L} \mathbf{x}^2 + \sum_{k=2}^{p-1} \mu_k \mathbf{x}^{k+1} + \sum_{k=2}^{p-1} \mu_k \mathbf{Q}^k \mathbf{d}^k + \mu_1 \mathbf{L} \mathbf{Q}^1 \mathbf{d}^1 + \sum_{k=1}^{p-1} \mu_{k+1} \dot{\mathbf{Q}}^k \hat{\mathbf{d}}^k - \sum_{k=1}^{p-1} \mu_{k+1} \mathbf{Q}^k (\mathbf{Q}^k)^\top \mathbf{Q}^k \mathbf{e}_d^k \\ &= -\mathbf{s} - \left(\text{diag} \left(\frac{\beta_i}{\|\mathbf{s}_i\| + \mu_i(t)} \right) \otimes \mathbf{I}_d \right) \mathbf{s} + \boldsymbol{\xi}(t),\end{aligned}\quad (4)$$

where $\boldsymbol{\xi}(t) = \sum_{k=2}^{p-1} \mu_k \mathbf{Q}^k \mathbf{d}^k + \mu_1 \mathbf{L} \mathbf{Q}^1 \mathbf{d}^1 - \sum_{k=1}^{p-1} \mu_{k+1} \mathbf{Q}^k (\mathbf{Q}^k)^\top \mathbf{Q}^k \mathbf{e}_d^k$ is a bounded unknown vector.

By considering the Lyapunov function $V = \frac{1}{2} \mathbf{s}^\top \mathbf{s} + \frac{1}{2} \sum_{i=1}^n \frac{1}{\gamma_i} (\beta_i - \beta_{\max})^2$, where $\beta_{\max} > \sup_{t \geq 0} \|\boldsymbol{\xi}\|$. Observe that V is positive definite, radially unbounded with regard to $\text{vec}(\mathbf{s}, \boldsymbol{\beta} - \mathbf{1}_n \beta_{\max})$, and bounded by two class- \mathcal{K}_∞ functions $V_1 = \frac{\alpha_1}{2} \mathbf{s}^\top \mathbf{s} + \frac{1}{2 \max_i \gamma_i} \sum_{i=1}^n (\beta_i - \beta_{\max})^2$, and $V_2 = \frac{\alpha_2}{2} \mathbf{s}^\top \mathbf{s} + \frac{1}{2 \min_i \gamma_i} \sum_{i=1}^n (\beta_i - \beta_{\max})^2$ with $0 < \alpha_1 < 1 < \alpha_2$. We have $\dot{V} \leq -\|\mathbf{s}\|^2 - \beta_{\max} \|\mathbf{s}\|_1 \leq 0$. By similar reasoning as in the proof of Thm. 6(ii), we have $\int_0^\infty \|\mathbf{s}(\tau)\|^2 d\tau$ is finite, $\lim_{t \rightarrow +\infty} \mathbf{s}(t) = \mathbf{0}_{dn}$, and $\lim_{t \rightarrow +\infty} \beta_i(t)$ exists and is finite.

Finally, in Eqn. (3), we may rewrite

$$\frac{d^{p-1}}{dt^{p-1}} \mathbf{x}^1 = -\mu_1 \mathbf{L} \mathbf{x}^1 - \sum_{k=1}^{p-2} \mu_{k+1} \frac{d^k \mathbf{x}^1}{dt^k} + \boldsymbol{\psi}, \quad (5)$$

where $\boldsymbol{\psi} = \mathbf{s} - \sum_{k=1}^{p-1} \mu_{k+1} \mathbf{Q}^k \mathbf{e}_d^k$ is a bounded, vanishing input to the nominal $(p-1)$ -th order matrix-weighted consensus system

$$\dot{\mathbf{x}}^k = \mathbf{x}^{k+1}, \quad k = 1, \dots, p-2, \quad (6a)$$

$$\dot{\mathbf{x}}^{p-1} = -\mu_1 \mathbf{L} \mathbf{x}^1 - \mu_2 \mathbf{x}^2 - \dots - \mu_{p-1} \mathbf{x}^{p-1}. \quad (6b)$$

Taking the variable transformation $\mathbf{y}^1 = \mathbf{V}^\top \mathbf{x}^1$, we have

$$\frac{d^{p-1}}{dt^{p-1}} \mathbf{y}^1 = -\mu_1 \boldsymbol{\Lambda} \mathbf{y}^1 - \sum_{k=1}^{p-2} \mu_{k+1} \frac{d^k \mathbf{y}^1}{dt^k} + \mathbf{V}^\top \boldsymbol{\psi}. \quad (7)$$

Let $y_{q,j+1} = \frac{d^j}{dt^j} \mathbf{v}_q^\top \mathbf{x}^1$, $q = l+1, \dots, dn, j = 0, \dots, p-2$, it follows that

$$\frac{d}{dt} \begin{bmatrix} y_{k,1} \\ \vdots \\ y_{k,p-1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0}_{p-2} & \mathbf{I}_{p-2} \\ -\mu_1 \lambda_k & -\boldsymbol{\mu}_{[2:p-1]}^\top \end{bmatrix}}_{:=\mathbf{A}_k} \begin{bmatrix} y_{k,1} \\ \vdots \\ y_{k,p-1} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{0}_{p-2} \\ \mathbf{v}_q^\top \boldsymbol{\psi} \end{bmatrix}}_{:=\boldsymbol{\varphi}_k(t)}$$

where $\boldsymbol{\mu}_{[2:p-1]}^\top = [\mu_2, \dots, \mu_{p-1}]$. As the matrix \mathbf{A}_k has characteristic polynomials $Q_{\lambda_k}(s)$, which is Hurwitz, and $\boldsymbol{\varphi}_k(t)$ is a vanishing input, it follows from Lemma 1 that $\lim_{t \rightarrow +\infty} y_{k,j} = 0$. Since $y_{k,1}$, $k = l+1, \dots, dn$, are state variables corresponding to the disagreement space $\ker(\mathbf{L})^\perp$, it follows that $\lim_{t \rightarrow +\infty} \mathbf{L}\mathbf{x}^1 = \mathbf{0}_{dn}$. Now, consider $y_{k,j+1} = \frac{d^j}{dt^j} \mathbf{v}_q^\top \mathbf{x}^1$, $q = 1, \dots, l, j = 0, \dots, p-2$, it follows that

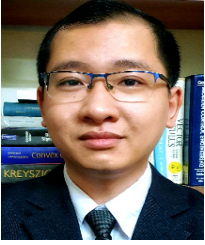
$$\frac{dy_{k,1}}{dt} = y_{k,2}, \quad (8a)$$

$$\frac{d^{p-2}y_{k,2}}{dt^{p-2}} = -\sum_{k=1}^{p-3} \mu_{k+1} \frac{d^k y_{k,2}}{dt^k} + \mathbf{v}_q^\top \boldsymbol{\psi}(t). \quad (8b)$$

It follows from Eqn. (8)(b), the assumption that $Q_0(s)$ is Hurwitz, and Lemma 1 that $\lim_{t \rightarrow +\infty} y_{k,j+1}(t) = 0$, $\forall j = 1, \dots, p-2$, and $\forall k = 1, \dots, l$, and $y_{k,1}(t) = \int_0^t y_{k,2}(\tau) d\tau$.

As we have shown that $\lim_{t \rightarrow +\infty} y_{k,j+1}(t) = 0$, $k = l, \dots, dn, j = 1, \dots, p-2$, it follows that $\dot{\mathbf{x}}^k(t) \rightarrow \mathbf{0}_{dn}$, $k = 1, \dots, p-1$. Thus, $\lim_{t \rightarrow +\infty} (\mathbf{x}^k(t) + \mathbf{Q}^{k-1} \mathbf{d}^{k-1}) = \mathbf{0}_{dn}$, $k = 2, \dots, p$. The proof is completed.

AUTHOR BIOGRAPHY



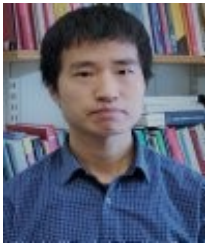
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